CENG114 Lecture Notes

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Contents

%includeCENG114-03

1 Review Session 2

1.1 Example 1

i.

$$\int_0^2 \int_0^2 k(x+y)dxdy = 1$$
$$k\int_0^2 \left[\frac{x^2}{2} + xy\right]_0^2 dy = 1$$
$$2k\int_0^2 [1+y]dy = 1$$
$$2k\left[y + \frac{y^2}{2}\right]_0^2 = 1$$
$$k = \frac{1}{8}$$

ii.

$$f_X(x) = \int_0^2 \frac{1}{8}(x+y)dy$$
$$= \frac{1}{8}\left[xy + \frac{y^2}{2}\right]_0^2$$
$$= \frac{1}{4}(x+1)$$
$$f_Y(y) = \frac{1}{4}(y+1)$$

iii.

$$E[X] = \int_0^2 x\frac{1}{4}(x+1)dx$$
$$= \frac{1}{4}\int_0^2 x^2 + xdx$$
$$= \frac{1}{4}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^2$$
$$= \frac{7}{6}$$
$$E[X^2] = \int_0^2 x^2\frac{1}{4}(x+1)dx$$
$$= \frac{1}{4}\int_0^2 x^3 + x^2dx$$
$$= \frac{1}{4}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^2$$
$$= \frac{5}{3}$$
$$\text{var}(X) = \frac{5}{3} - \left(\frac{7}{6}\right)^2 = \frac{11}{36}$$

Values for Y are the same given the marginal PMFs have the same form.

iv.

$$
\begin{aligned}
E[XY] &= \int_0^2 \int_0^2 xy\frac{1}{8}(x+y)dxdy \\
&= \frac{1}{8}\int_0^2 \int_0^2 x^2y + xy^2 dxdy \\
&= \frac{1}{8}\int_0^2 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_0^2 dy \\
&= \frac{1}{8}\int_0^2 \left[\frac{8y}{3} + \frac{4y^2}{2}\right] dy \\
&= \frac{1}{8}\left[\frac{8y^2}{3\times 2} + \frac{4y^3}{2\times 3}\right]_0^2 \\
&= \frac{4}{3} \\
\mathrm{cov}(X,Y) &= E[XY] - E[X]E[Y] \\
&= \frac{4}{3} - \frac{7}{6}\times\frac{7}{6} \\
&= -\frac{1}{36}
\end{aligned}
$$

1.2 Example 2

i.

From Normal Table, we find that $P(Z < 1) = 0.8413$.

$$
\begin{aligned}
z &= \frac{x-\mu}{\sigma} = 1 \\
\frac{200 - 159.2}{\sigma} &= 1 \\
\sigma = 40.8
\end{aligned}
$$

ii.

$$\begin{aligned}
P(Z > z) &= 0.9 \\
1 - P(Z < z) &= 0.9 \\
P(Z < z) &= 0.1 \\
P(Z > -z) &= 0.1 \\
1 - P(Z < -z) &= 0.1 \\
P(Z < -z) &= 0.9 \\
-z &= 1.28 \\
z &= -1.28 \\
\frac{x - 159.2}{40.8} &= -1.28 \\
x &= 106.98
\end{aligned}$$

iii.

$$\begin{aligned}
P(\mu + 0.5\sigma < X < \mu + 2\sigma) &= P(0.5 < Z < 2) \\
&= P(Z < 2) - P(Z < 0.5) \\
&= 0.9773 - 0.6915 \\
&= 0.2858
\end{aligned}$$

iv.

$$\begin{aligned}
P(X < 170) &= P(Z < \frac{170 - 159.2}{40.8}) \\
&= P(Z < 0.26) \\
&= 0.6026
\end{aligned}$$

The number of adults with X ¡ 170 has distribution $Y\ B(5, 0.6026)$

$$P(Y = 3) = {}^5C_3 \times 0.6026^3 \times (1 - 0.6026)^2 = 0.3456$$

1.3 Example 3

i.

$$
\begin{aligned}
P(k \leq 3) &= P(k=0) + P(k=1) + P(k=2) + P(k=3) \\
&= e^{-2\times 4}(1 + 8 + \frac{8^2}{2!} + \frac{8^3}{3!}) \\
&= 0.0424
\end{aligned}
$$

ii.

$$
\begin{aligned}
P(X > 2) &= 1 - P(k=0) - P(k=1) - P(k=2) \\
&= 1 - e^{-2\times 3}(1 + 6 + \frac{6^2}{2!}) \\
&= 0.938
\end{aligned}
$$

iii.

Note that this is for 5 meters!

$$
\begin{aligned}
\lambda\tau &= 2 \times 5 \times 10^3 = 10^4 \\
P(9900 < x < 10250) &= P(x < 10250) - P(x < 9900) \\
&= P(Z < \frac{10250 - 10000}{\sqrt{10000}}) - P(Z < \frac{9900 - 10000}{\sqrt{10000}}) \\
&= P(Z < 2.5) - P(Z < -1) \\
&= 0.9938 - (1 - P(Z < 1)) \\
&= 0.9938 - (1 - 0.8413) \\
&= 0.8351
\end{aligned}
$$

iv.

$$
\begin{aligned}
P(k=0) &= 0.9 = e^{-\lambda L} \\
-5\lambda &= \ln 0.9 \\
\lambda &= 0.0211 \text{ flaws/mm}
\end{aligned}
$$

v.

$$\begin{aligned} p &= e^{-0.0211} - 0.979 \\ P(k) &= {}^{10}C_k \times 0.979^k \times (1 - 0.979)^{10-k} \\ P(3) &= {}^{10}C_3 \times 0.979^3 \times (1 - 0.979)^{10-3} \\ &= 2.028 \times 10^{-10} \end{aligned}$$

1.4 Example 4

i. Rate constant $= \frac{\ln 2}{10} = 0.0693 \text{ min}^{-1}$

ii.

$$\int_0^t A_0 \lambda e^{-\lambda t} dt = A$$

$$1 - e^{-\lambda t} = 0.75$$

$$e^{-\lambda t} = 0.25$$

$$t = 20 \text{ mins}$$

iii.

$$B : C = 2 : 1$$

$B = \frac{2}{3}A$.

$$\begin{aligned} A &= A_0(1 - e^{-\lambda t}) \\ &= A_0(1 - e^{-0.0693 \times 15}) \\ &= 0.6464 A_0 \end{aligned}$$

$$\begin{aligned}\text{Weight} &= (A - A_0) + B \\ &= (1 - 0.6464)A_0 + \frac{2}{3}0.6464A_0 \\ &= 0.785A_0\end{aligned}$$

So weight after 15 mins is 78.5% of initial.