CENG114 Lecture Notes

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Contents

1 Lecture 06

1.1 Slide 4

$$
\begin{aligned}
p_{X,Y}(x,y) &= \frac{1}{24} \\
p_X(x) &= \sum_{y=1}^{6} \frac{1}{24} = \frac{1}{4} \\
p_Y(y) &= \sum_{x=1}^{4} \frac{1}{24} = \frac{1}{6}
\end{aligned}
$$

1.2 Slide 7

$A = \{1, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}$

$P(A) = \frac{12}{20} = \frac{3}{5}$

$$
\begin{aligned}
p_{X|A}(x) &= \frac{P(X = x \cap A)}{P(A)} \\
&= \frac{1}{20} / \frac{3}{5} \\
&= \frac{1}{12}
\end{aligned}
$$

1.3 Slide 8

$$
\begin{aligned}
p_X(x) &= (1-p)^{x-1}p \\
P(Y) &= \sum_{x=0}^{\infty}(1-p)^{2x+1}p \\
&= p(1-p)\sum_{x=0}^{\infty}(1-p)^{2x} \\
&= p(1-p)\frac{1}{1-(1-p)^2} \\
&= \frac{1-p}{2-p} \\
p_{X|Y}(x) &= \frac{P(X=x\cap Y)}{P(Y)} \\
&= \frac{(1-p)^{x-1}p(2-p)}{1-p} \\
&= (1-p)^{x-2}p(2-p)
\end{aligned}
$$

1.4 Slide 10

For $x = 1, 2, 3, 4$, the result can come from either the 4 or 6-sided die.

$$
\begin{aligned}
p_X(x) &= \frac{1}{3}\frac{1}{4} + \frac{2}{3}\frac{1}{6} \\
&= \frac{7}{36}
\end{aligned}
$$

For $x = 5, 6$, the result can come from only 6-sided die.

$$
\begin{aligned}
p_X(x) &= \frac{2}{3}\frac{1}{6} \\
&= \frac{1}{9}
\end{aligned}
$$

$$
p_X(x) = \begin{cases} \frac{7}{36} & x = 1, 2, 3, 4 \\ \frac{1}{9} & x = 5, 6 \\ 0 & \text{otherwise} \end{cases}
$$

1.5 Slide 12

$$\begin{aligned} E[X] &= E[X|Y=4]P(Y=4) + E[X|Y=6]P(Y=6) \\ &= 2.5 \times \frac{1}{3} + 3.5 \times \frac{2}{3} \\ &= 3.17 \end{aligned}$$

Validate using the actual PMF.

1.6 Slide 20

i. Using the axiom of normalization, we have

$$\begin{aligned} \int_0^1 \int_0^1 kxydxdy &= k \left.\frac{x^2}{2}\right|_0^1 \left.\frac{y^2}{2}\right|_0^1 \\ k &= 4 \end{aligned}$$

ii.

$$\begin{aligned} P(X < 0.5, Y < 0.5) &= \int_0^{0.5} \int_0^{0.5} 4xydxdy \\ &= 4 \left.\frac{x^2}{2}\right|_0^{0.5} \left.\frac{y^2}{2}\right|_0^{0.5} \\ &= 0.0625 \end{aligned}$$

iii.

$$\begin{aligned} E[XY] &= \int_0^1 \int_0^1 xy4xydxdy \\ &= 4 \left.\frac{x^3}{3}\right|_0^1 \left.\frac{y^3}{3}\right|_0^1 \\ &= \frac{4}{9} \end{aligned}$$

iv.

$$\begin{aligned} f_X(x) &= \int_0^1 4xy\,dy \\ &= 4x \left.\frac{y^2}{2}\right|_0^1 \\ &= 2x \end{aligned}$$

1.7 Slide 25

$$E[X] = \frac{1}{\lambda} = 50$$

$$\lambda = \frac{1}{50}$$

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x)$$

$$= \frac{1}{\sqrt{2\pi(0.1x)^2}} e^{-\frac{y^2}{2(0.1x)^2}} \frac{1}{50} e^{-\frac{x}{50}}$$

1.8 Slide 29

$$f_{Y|1}(y|1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}}$$

$$f_{Y|-1}(y|-1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2}}$$

$$P(S=1|Y) = \frac{P(S=1) f_{Y|1}(y|1)}{P(S=1) f_{Y|1}(y|1) + P(S=-1) f_{Y|-1}(y|-1)}$$

$$= \frac{p\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}}}{p\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}} + (1-p)\frac{1}{\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2}}}$$

$$= \frac{pe^{-\frac{(y-1)^2}{2}}}{pe^{-\frac{(y-1)^2}{2}} + (1-p)e^{-\frac{(y+1)^2}{2}}}$$

$$= \frac{p}{p + (1-p)e^{-\frac{(y+1)^2}{2} + \frac{(y-1)^2}{2}}}$$

$$= \frac{p}{p + (1-p)e^{-2y}}$$

1.9 Slide 31

$$f_{X,Y}(x,y) = \frac{12}{11}(x^2 + xy + y^2)$$

$$f_X(x) = \int_0^1 \frac{12}{11}(x^2 + xy + y^2)dy$$

$$= \frac{12}{11}\left[x^2 y + \frac{xy^2}{2} + \frac{y^3}{3}\right]_0^1$$

$$= \frac{12}{11}(x^2 + \frac{x}{2} + \frac{1}{3})$$

$$f_Y(y) = \int_0^1 \frac{12}{11}(x^2 + xy + y^2)dx$$

$$= \frac{12}{11}(y^2 + \frac{y}{2} + \frac{1}{3})$$

$$f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$$

$$f_{X,Y}(x,y) = \frac{4}{9}(xy + x + y + 1)$$

$$f_X(x) = \int_0^1 \frac{4}{9}(xy + x + y + 1)dy$$

$$= \frac{4}{9}\left[x\frac{y^2}{2} + xy + \frac{y^2}{2} + y\right]_0^1$$

$$= \frac{2}{3}(x + 1)$$

$$f_Y(y) = \int_0^1 \frac{4}{9}(xy + x + y + 1)dx$$

$$= \frac{2}{3}(y + 1)$$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

1.10 Slide 34

$$
\begin{aligned}
E[XY] &= \int_0^1 \int_0^1 \frac{12}{11} xy(x^2 + xy + y^2) dx dy \\
&= \frac{12}{11} \int_0^1 \int_0^1 (x^3 y + x^2 y^2 + x y^3) dx dy \\
&= \frac{12}{11} \int_0^1 \left[\frac{x^4}{4} y + \frac{x^3 y^2}{3} + \frac{x^2 y^3}{2} \right]_0^1 dy \\
&= \frac{12}{11} \int_0^1 \frac{1}{4} y + \frac{y^2}{3} + \frac{y^3}{2} dy \\
&= \frac{12}{11} \left[\frac{1}{4} \frac{y^2}{2} + \frac{y^3}{3 \times 3} + \frac{y^4}{2 \times 4} \right]_0^1 \\
&= \frac{13}{33} \\
E[X] &= \int_0^1 \frac{12}{11} x(x^2 + \frac{x}{2} + \frac{1}{3}) dx \\
&= \frac{12}{11} \left[\frac{x^4}{4} + \frac{x^3}{2 \times 3} + \frac{x^2}{3 \times 2} \right]_0^1 \\
&= \frac{7}{11} \\
E[Y] &= \frac{7}{11} \\
\operatorname{cov}(X, Y) &= E[XY] - E[X]E[Y] \\
&= \frac{13}{33} - \frac{7}{11} \frac{7}{11} \\
&= -\frac{4}{363}
\end{aligned}
$$

$$E[X^2] = \int_0^1 \frac{12}{11} x^2 (x^2 + \frac{x}{2} + \frac{1}{3}) dx$$

$$= \frac{12}{11} \left[\frac{x^5}{5} + \frac{x^4}{2 \times 4} + \frac{x^3}{3 \times 3} \right]_0^1$$

$$= \frac{157}{330}$$

$$\sigma_X = \sqrt{\frac{157}{330} - \left(\frac{7}{11} \right)^2}$$

$$= 0.266 = \sigma_Y$$

$$\rho_{X,Y} = \frac{-4/363}{0.266 \times 0.266}$$

$$= -0.156$$