# CENG114 Lecture Notes

Shyue Ping Ong

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## Contents

# 1 Lecture 05

## 1.1 Slide 5

$$\begin{aligned} E[X] &= p \times 1 + (1-p) \times 0 = p \\ E[X^2] &= p \times 1^2 + (1-p) \times 0^2 = p \\ var(X) &= E[X^2] - E[X]^2 = p - p^2 = p(1-p) \end{aligned}$$

## 1.2 Slide 6

A binomial RV can be though of a linear sum of the results of $n$ Bernoulli trials.

$X = Y_1 + Y_2 + ... + Y_n$

$$\begin{aligned} E[X] &= E[Y_1] + E[Y_1] + ... + E[Y_n] = np \\ var(X) &= var(Y_1) + var(Y_2) + ... + var(Y_n) = np(1-p) \end{aligned}$$

## 1.3 Slide 8

$X\ B(20, 0.01)$

$$\begin{aligned} E[X] &= 20 \times 0.01 = 0.2 = \mu \\ var(X) &= 20 \times 0.01 \times (1 - 0.01) = 0.198 = \sigma^2 \\ \sigma &= 0.445 \end{aligned}$$

We need $p_X(k > \mu + 3\sigma) = p_X(k > 1.535)$.

$$\begin{aligned} p_X(k > 1.535) &= 1 - p_X(k \leq 1.535) \\ &= 1 - p_X(k = 0) - p_X(k = 1) \\ &= 1 - 0.99^{20} -^{20} C_1 \times 0.01^1 \times 0.99^1 9 \\ &= 0.0169 \end{aligned}$$

## 1.4 Slide 9

$$\begin{aligned}\sum p_X(k) &= p + (1-p)p + (1-p)^2 p + ... \\ &= p[1 + (1-p) + (1-p)^2 - ...] \\ &= p\frac{1}{1-(1-p)} \\ &= 1\end{aligned}$$

## 1.5 Slide 13

$\sum p_X(k) = e^{-\lambda\tau}[1 + \frac{(\lambda\tau)^1}{1!} + \frac{(\lambda\tau)^2}{2!}...]$

From Taylor series, we know that

$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!}...$

$\sum p_X(k) = e^{-\lambda\tau}e^{\lambda\tau} = 1$

## 1.6 Slide 15

$\lambda = 2.5/mm^3$

a. $p_X(k \geq 1) = 1 - p_X(k=0) = 1 - e^{-2.5\times 1} = 0.918$

b.

$$\begin{aligned}\lambda\tau &= 2.5 \times 5 = 12.5 \\ p_X(k \geq 4) &= 1 - p_X(k < 4) \\ &= 1 - p_X(k=0) - p_X(k=1) - -p_X(k=2) - p_X(k=3) \\ &= 1 - e^{-12.5}(1 + 12.5 + \frac{12.5^2}{2!} + \frac{12.5^3}{3!}) \\ &= 0.9984\end{aligned}$$

c.

$$
\begin{aligned}
\lambda\tau &= 2.5 \times V \\
p_X(k \geq 1) &= 1 - p_X(k=0) > 0.99 \\
1 - e^{-2.5V} &> 0.99 \\
e^{-2.5V} &< 0.01 \\
V &> 1.842mm^3
\end{aligned}
$$

## 1.7 Slide 19

$$
\begin{aligned}
1 - e^{-}\lambda t_{0.5} &= 0.5 \\
-\lambda t_{0.5} &= \ln(0.5)
\end{aligned}
$$

$$
t_{0.5} = \frac{\ln 2}{\lambda}
$$

## 1.8 Slide 20

$$
\begin{aligned}
E[X] &= \frac{1}{\lambda} = 60 \\
\lambda &= \frac{1}{60} \\
P(t < 30) &= 1 - e^{-\frac{1}{60}30} = 0.393
\end{aligned}
$$

$$
\begin{aligned}
P(t < 120) &= 1 - e^{-\frac{1}{60}120} = 0.865 \\
P(120 < t < 150) &= P(t < 150) - P(t < 120) \\
&= 1 - e^{-\frac{1}{60}150} - 0.865 \\
&= 0.0532 \\
P(t < 150 | t > 120) &= \frac{P(120 < t < 150)}{P(t > 120)} \\
&= \frac{0.0532}{1 - 0.865} = 0.393
\end{aligned}
$$

## 1.9 Slide 21

Amount of $C^{14}$ decayed $= 32 - 12 = 20$ Fraction of $C^{14}$ decayed over life $= \frac{20}{32} = 0.625$

$$\begin{aligned} t_{0.5} &= \frac{\ln 2}{\lambda} \\ \lambda &= \frac{\ln 2}{5730} \end{aligned}$$

Since 0.625 has decayed over the life, we have:

$$\begin{aligned} P(t < T) &= 0.625 \\ 1 - e^{-\frac{\ln 2}{5730}T} &= 0.625 \\ -\frac{\ln 2}{5730}T &= \ln 0.375 \\ T &= -\frac{\ln 0.375}{\ln 2}5730 = 8108 \end{aligned}$$

## 1.10 Slide 28

i. Note that $P(Z < 1.64) = 0.945$ and $P(Z < 1.65) = 0.955$. 0.95 is halfway between 0.945 and 0.955. So we can approximate that $z = 1.645$.

ii.

$$\begin{aligned} P(-z < Z < z) &= 0.95 \\ P(Z < z) - P(Z < -z) &= 0.95 \\ P(Z < z) - P(Z > z) &= 0.95 \\ P(Z < z) - (1 - P(Z < z)) &= 0.95 \\ 2P(Z < z) &= 1.95 \\ P(Z < z) &= 0.975 \end{aligned}$$

From table, we get $P(Z < 1.96) = 0.975$. So $z = 1.96$.

## 1.11 Slide 30

$$\begin{aligned}
P(X > 300) &= P(Z > \frac{300-200}{50}) \\
&= P(Z > 2) \\
&= 1 - P(Z < 2) \\
&= 1 - 0.9772 = 0.0228
\end{aligned}$$

$$\begin{aligned}
P(100 < X < 250) &= P(\frac{100-200}{50} < Z < \frac{250-200}{50}) \\
&= P(-2 < Z < 1) \\
&= P(Z < 1) - P(Z < -2) \\
&= P(Z < 1) - P(Z > 2) \\
&= P(Z < 1) - (1 - P(Z < 2)) \\
&= 0.8413 - (1 - 0.9772) = 0.8185
\end{aligned}$$

## 1.12 Slide 35

$S_n \sim N(80, \frac{10^2}{30})$

$$\begin{aligned}
P(|S_n - \mu| > 0.05\mu) &= P(S_n < 0.95\mu) + P(S_n > 1.05\mu) \\
&= P(Z < \frac{0.95\mu - \mu}{\sqrt{\frac{10^2}{30}}}) + P(Z > \frac{1.05\mu - \mu}{\sqrt{\frac{10^2}{30}}}) \\
&= P(Z < -2.19) + P(Z > 2.19) \\
&= 2P(Z > 2.19) \\
&= 2(1 - 0.9857) = 0.0285
\end{aligned}$$

## 1.13 Slide 36

$$\begin{aligned}
P(|S_n - \mu| > 0.01\mu) &< 0.05 \\
P(Z < \frac{0.99\mu - \mu}{\sqrt{\frac{10^2}{n}}}) + P(Z > \frac{1.01\mu - \mu}{\sqrt{\frac{10^2}{n}}}) &< 0.05 \\
P(Z < -0.001\mu\sqrt{n}) + P(Z > 0.001\mu\sqrt{n}) &< 0.05 \\
2P(Z > 0.001\mu\sqrt{n})) &< 0.05 \\
2(1 - P(Z < 0.001\mu\sqrt{n})) &< 0.05 \\
P(Z < 0.001\mu\sqrt{n}) &> 0.975 \\
0.001\mu\sqrt{n} &> 1.96 \\
n &> 600.25
\end{aligned}$$

## 1.14 Slide 40

$$\begin{aligned}
P(X > 1000) &\sim P(Z > \frac{1000 - 0.5 - 0.095 \times 10000}{\sqrt{10000 \times 0.095 \times (1 - 0.095)}}) \\
&= P(Z > 1.688) \\
&= 1 - P(Z < 1.688) \\
&= 0.0457
\end{aligned}$$