CENG114 Lecture Notes

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Contents

1 Lecture 02

1.1 Slide 4

Q1: $6 \times 4 = 24$

Q2:

To form a subset, we make a decision as to whether to select each integer.

For example, the empty set occurs when we make a decision to not to select all integers.

The set $\{1\}$ is obtained when we make a decision to select 1, but none of the other integers.

Each integer can either be selected, or not selected, i.e., 2 choices. So the total number of ways to make this decision is

$2^{10} = 1024$

1.2 Slide 5

For permutation, we have

$(1,2)$ $(1,3)$ $(2,1)$ $(2,3)$ $(3,1)$ $(3,2)$

Six different ways. We can also reason it as we choose one ball, followed by choosing another ball. For the first ball, we have three choices (1, 2, or 3). For the second ball, we are left with only 2 choices. So total number of ways is $3 \times 2 = 6$

For combination, we have

$(1,2)$ $(1,3)$ $(2,3)$

Three ways.

1.3 Slide 6

$$\begin{aligned} {}^{n}P_{k} &= n.(n-1).(n-2)...(n-(k-1)) \\ &= \frac{n.(n-1).(n-2)...1}{(n-k).(n-k-1).(n-k-2)...1} \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

1.4 Slide 7

Q1: Special case of k-permutation.

${}^{n}P_{n} = \frac{n!}{(n-n)!} = n!$

Remember that $0! = 1$.

Q2: We have $26 + 26 + 10 = 62$ characters.

For password of 6 characters, we have ${}^{62}P_6$.

For password of 7 characters, we have ${}^{62}P_7$.

For password of 8 characters, we have ${}^{62}P_8$.

Total number of ways $= {}^{62}P_6 +^{62} P_7 +^{62} P_8 = 1.38 \times 10^{14}$.

If repetition is allowed, we have 62 choices for each character.

Total number of ways $= 62^6 + 62^7 + 62^8 = 2.22 \times 10^{14}$.

1.5 Slide 8

i. We either have Red-Black, or Black-Red - two possibilities. For each set of color, we have 10 choices for the first ball, 9 choices for second, ..., i..e, ${}^{10}P_1 0 = 10!$ possibilities.

Total number of ways $= 2 \times 10! \times 10! = 2.63 \times 10^{13}$

i. For the numbers, we have 10! arrangements of numbers. For each number, we have 2 ways (BR or RB).

Total number of ways $= 10! \times 2^{10} = 3.72 \times 10^9$

1.6 Slide 10

${}^{n}C_{k} = \frac{{}^{n}P_{k}}{k!} = \frac{n!}{(n-k)!k!}$

1.7 Slide 11

Q1: ${}^{4}C_1 = \frac{4!}{3!1!} = 4$

Q2: ${}^{30}C_5 \times 25 \times 24 = \frac{30!}{25!5!} \times 25 \times 24 = 85503600$

What if we choose the President and VP first?

$30 \times 29 \times^{28} C_5 = 30 \times 29 \times \frac{28!}{23!5!} = 85503600$

1.8 Slide 13

Total count

$$\begin{aligned} &= \frac{n!}{(n-n_1)!n_1!}\frac{(n-n_1)!}{(n-n_1-n_2)!n_2!}\cdots\frac{n_r!}{(n-n_1-n_2-...-n_{r-1})!n_r!} \\ &= \frac{n!}{n_1!n_2!...n_r!} \end{aligned}$$

1.9 Slide 14

Number of ways to partition 50 students into groups of 10 $= \frac{50!}{10!10!10!10!10!}$

Ways to assign a senior to each group $= 5!$

So total number of ways $= 5! \times \frac{50!}{10!10!10!10!10!} = 5.80 \times 10^{33}$

1.10 Slide 16

Total count $= \frac{9!}{3!3!3!} = 1680$

1.11 Slide 18

Q1. Number of combinations of numbers $= {}^{42}C_6 = 5245786$

So probability (only one set of winning numbers) $= \frac{1}{5245786}$

Q2. Number of ways to get the 5 of 6 numbers $= {}^{6}C_5$. Special number is fixed.

Probability of winning second prize $= \frac{{}^{6}C_5}{{}^{42}C_6} = \frac{3}{2622893}$

1.12 Slide 19

Ways to get 5: 1 + 1 + 3, 1 + 2 + 2.

For 1 + 1 + 3, the 3 can occur on the first, second or third dice. So 3 permutations.

For 1 + 2 + 2, the 1 can occur on the first, second or third dice. So 3 permutations.

$P(5) = \frac{3+3}{6\times6\times6} = \frac{1}{36}$

2 Lecture 03

2.1 Slide 3

$P(\text{roll is 6}|\text{roll is even}) = \frac{1}{3}$

2.2 Slide 4

Normalization: $P(\Omega|B) = \frac{P(\Omega\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

Additivity:

$$\begin{aligned}
P(A_1 \cup A_2|B) &= \frac{P((A_1 \cup A_2) \cap B)}{P(B)} \\
&= \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)} \\
&= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} \\
&= P(A_1|B) + P(A_2|B)
\end{aligned}$$

2.3 Slide 5

Intuitive approach: If first toss is a H, only way there are more T than H is for the next two tosses to be T.

P(more tails than heads in 3 tosses— first toss is heads) = 1/2 * 1/2 = 1/4

Mathematical approach:

Let X_n denote a result of X on the nth toss.

$P(T_2T_3 \cap H_1) = 0.5 \times 0.5 \times 0.5 = 0.125$

$P(H_1) = 0.5$

$P(T_2T_3|H_1) = \frac{P(T_2T_3\cap H_1)}{P(H_1)} = 0.25$

2.4 Slide 7

Number of picture cards $= 4 \times 4 = 16$ Number of cards left $= 52 - 3 = 49$

$$\begin{aligned} P(\text{Dealer picture} \cap \text{Cards shown}) &= \frac{1 \times 1 \times 1 \times 16}{52 \times 51 \times 50 \times 49} \\ P(\text{Cards shown}) &= \frac{1 \times 1 \times 1}{52 \times 51 \times 50} \\ P(\text{Dealer picture}|\text{Card shown}) &= \frac{P(\text{Dealer picture} \cap \text{Cards shown})}{P(\text{Cards shown})} = \frac{16}{49} \end{aligned}$$

If you are showing two picture cards,

Number of picture cards left $= 16 - 2 = 14$

$$\begin{aligned} P(\text{Dealer picture} \cap \text{Cards shown}) &= \frac{1 \times 1 \times 1 \times 14}{52 \times 51 \times 50 \times 49} \\ P(\text{Dealer picture}|\text{Card shown}) &= \frac{P(\text{Dealer picture} \cap \text{Cards shown})}{P(\text{Cards shown})} = \frac{14}{49} \end{aligned}$$

2.5 Slide 8

(Draw grid.)

$P(X + Y = 7 \cap \max(X, Y) = 4) = \frac{2}{36}$

$P(\max(X, Y) = 4) = \frac{7}{36}$

$P(X + Y = 7 | \max(X, Y) = 4) = \frac{2}{7}$

2.6 Slide 10

$$\begin{aligned} P(\text{Flush}) &= P(A_1)P(A_2|A_1)P(A_3|A_2 \cap A_1)P(A_4|A_3 \cap A_2 \cap A_1)P(A_5|A_4 \cap A_3 \cap A_2 \cap A_1) \\ &= \frac{52}{52}\frac{12}{51}\frac{11}{50}\frac{10}{49}\frac{9}{48} \\ &= 0.00198 \end{aligned}$$

2.7 Slide 11

$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) + ... + P(A_n \cap B) \\ &= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + ... + P(B|A_n)P(A_n) \end{aligned}$$

2.8 Slide 12

$P(delayed) = 0.8 \times \frac{30}{365} + 0.4 \times \frac{365-30}{365} = 0.433$

2.9 Slide 15

Let A denote the event that the woman has breast cancer, and B denote the event that it is detected. First step in all probability problems is to write down the probabilities that the question gives you:

$P(B|A) = 0.8$

$P(B^C|A^C) = 0.8$

$P(A) = 0.12$

$$\begin{aligned} P(A|B) &= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^C)P(A^C)} \\ &= \frac{0.8 \times 0.12}{0.8 \times 0.12 + (1-0.8) \times (1-0.12)} \\ &= 0.353 \end{aligned}$$

Note that the probability of the woman having cancer, even with a positive test, is not 0.8, but 0.353. This is because the incidence of breast cancer is low, and there is a chance of false positive.

2.10 Slide 19

Let A, B, C and D denote the event that a sample contains organic, volatile and chlorinated, no pollutants respectively. Let X be the event that a signal is detected.

$P(X|A) = 0.9$

$P(X|B) = 0.99$

$P(X|C) = 0.95$

$P(X|D) = 0.01$

$P(A) = 0.2$

$P(B) = 0.27$

$P(C) = 0.13$

$P(D) = 1 - 0.2 - 0.27 - 0.13 = 0.4$

Total probability theorem.

$$
\begin{aligned}
P(X) &= P(X|A)P(A) + P(X|B)P(B) + P(X|C)P(C) + P(X|D)P(D) \\
&= 0.9 \times 0.2 + 0.99 \times 0.27 + 0.95 \times 0.13 + 0.01 \times 0.4 \\
&= 0.5748
\end{aligned}
$$

$$
\begin{aligned}
P(C|X) &= \frac{P(X|C)P(C)}{P(X)} \\
&= \frac{0.95 \times 0.13}{0.5748} \\
&= 0.215
\end{aligned}
$$

2.11 Slide 18

1. Without replacement,

$$
\begin{aligned}
P(A \cap B) &= \frac{4}{52}\frac{4}{51} = \frac{4}{663} \\
P(A) &= \frac{4}{52} = \frac{1}{13} \\
P(B) &= \frac{4}{52} = \frac{1}{13}
\end{aligned}
$$

Clearly, $P(A \cap B) \neq P(A)P(B)$.

2. With replacement,

$$
\begin{aligned}
P(A \cap B) &= \frac{4}{52}\frac{4}{52} = \frac{1}{169} \\
P(A) &= \frac{4}{52} = \frac{1}{13} \\
P(B) &= \frac{4}{52} = \frac{1}{13}
\end{aligned}
$$

Clearly, $P(A \cap B) = P(A)P(B)$.

2.12 Slide 21

We need to analyze each subsection.

$$
\begin{aligned}
P(C \to E \to B) &= 0.8 \times 0.9 = 0.72 \\
P(C \to F \to B) &= 0.95 \times 0.85 = 0.8075 \\
P(C \to B) &= 1 - P(C \to F \to B fails \cap C \to E \to B fails) \\
&= 1 - (1 - 0.72)(1 - 0.8075) \\
&= 0.9461
\end{aligned}
$$

Simplify diagram. Now we need to analyze $A \to B$.

$$
\begin{aligned}
P(A \to C \to B) &= 0.9 \times 0.9461 = 0.8515 \\
P(A \to D \to B) &= 0.75 \times 0.95 = 0.7125 \\
P(A \to B) &= 1 - P(A \to C \to B fails \cap A \to D \to B fails) \\
&= 1 - (1 - 0.8515)(1 - 0.7125) \\
&= 0.9573
\end{aligned}
$$

2.13 Slide 22

IMPORTANT: Note that probability of failure is given, not probability of success!

$$
\begin{aligned}
P(A \to Btop) &= (1 - 0.01)(1 - 0.01) = 0.9801 \\
P(A \to B) &= 1 - P(A \to Btopfails \cap A \to Bbottomfails) \\
&= 1 - (1 - 0.9801) \times 0.1 \\
&= 0.99801
\end{aligned}
$$

$$
\begin{aligned}
P(B \rightarrow C) &= 1 - P(B \rightarrow Btopfails \cap B \rightarrow Cbottomfails) \\
&= 1 - 0.1 \times 0.1 \\
&= 0.99
\end{aligned}
$$

$P(A \rightarrow C) = P(A \rightarrow B)P(B \rightarrow C) = 0.99801 \times 0.99 = 0.988$

2.14 Slide 23

Let H denote the number of heads.

Hard way:

$P(H \geq 1) = P(H = 1) + P(H = 2) + P(H = 3) + P(H = 4) + P(H = 5)$

To get one head, we have HTTTT, THTTT, ... Five positions to chose the one H. Each coin toss has a 0.5 probability of either H or T.

$P(H = 1) =^5 C_1 \times 0.5^5$

$P(H = 2) =^5 C_2 \times 0.5^5$

$P(H = 3) =^5 C_3 \times 0.5^5$

$P(H = 4) =^5 C_4 \times 0.5^5$

$P(H = 5) =^5 C_5 \times 0.5^5$

Easier way: The opposite of at least one H is there are no H.

$P(H \geq 1) = 1 - P(H = 0) = 1 - 0.5^5 = 0.96875$