Name: ______________________________ ID #: ______________________________

CENG 114 – Probability and Statistical Methods for Engineers

Professor Shyue Ping Ong

Mid-term Exam 2

Winter, Feb 27 2018

Instructions:

1. Write your name on the top right of this page.
2. Please answer **ALL** questions as far as possible.
3. You have a total of 80 mins exactly.
4. Write your answers in the space provided following the questions. If you need additional space, you may request for additional blank sheets of paper during the exam. If you use additional blank sheets, you should mark the top left of each sheet with your name, ID#, question number, and today's date, e.g., "John Smith, A1234567, Q1, Feb 27 2018".
5. Indicative maximum points for each sub-question are provided only as a reference. Actual grading points may vary.
6. Write neatly and legibly. Illegible answers will be considered as wrong.

Question 1 (20 points): ____________

Question 2 (20 points): ____________

Question 3 (20 points): ____________

Question 4 (20 points): ____________

Question 5 (20 points): ____________

This exam has a total of 7 pages, including the cover page.

Qn 1.

A joint probability distribution function (PDF) is given by:

$$f_{X,Y}(x,y) = \begin{cases} \frac{k}{xy} & 1 < x < 2, 1 < y < 2 \\ 0 & otherwise \end{cases}$$

Answer the following questions, giving your answers to 3 decimal places:

i. (5 points) Calculate the value of k that makes this a valid PDF.
ii. (4 points) Calculate $P(X < 1.5, Y < 1.5)$.
iii. (4 points) Derive the marginal PDF of X, $f_X(x)$.
iv. (3 points) Calculate E[XY].
v. (4 points) A third variable is defined as $U = X^2$. Derive the PDF of U.

Solutions

i. $\int_1^2 \int_1^2 \frac{k}{xy} dxdy = 1 \rightarrow k(lnx)|_1^2(lny)|_1^2 = 1, k = \frac{1}{(ln2)^2} = 2.081$

ii. $P(X < 1.5, Y < 1.5) = \int_1^{1.5} \int_1^{1.5} \frac{k}{xy} dxdy = \frac{1}{(ln2)^2}(ln1.5)^2 = 0.342$

iii. $f_x(x) = \int_1^2 \frac{k}{xy} dy = \frac{1}{x(ln2)}$

iv. $E[XY] = \int_1^2 \int_1^2 xy \frac{k}{xy} dxdy = k = \frac{1}{(ln2)^2} = 2.081$

v. For $1 < x < 2, 1 < u < 4,$

$$F_u(u) = \int_1^{\sqrt{u}} \frac{1}{xln2} dx = lnx|_1^{\sqrt{u}} \frac{1}{ln2} = \frac{ln\sqrt{u}}{ln2}$$

$$f_u(u) = \frac{\partial F_u(u)}{\partial u} = \frac{1}{ln2} \frac{1}{\sqrt{u}} \frac{1}{2} u^{-\frac{1}{2}} = \frac{1}{2uln2} \; for \; 1 < u < 4$$

Qn 2. A chemical engineering student is carrying out vapor deposition experiments. On any given day, he comes into the lab and has the choice of two machines X and Y. The thickness of the deposited chemical product (in nm) on machines X and Y follow Normal distributions, where X ~ *N*(1000, 625) and Y ~ *N*(980, 784). Answer the following questions, giving your answers to three decimal places.

i. (3 points) The deposited thickness meets requirements if it is between 950 and 1050 nm. If the student used machine X, what is the probability that he meets the requirement?

ii. (3 points) If the student used machine Y, what is the probability that he meets the requirement?

iii. (6 points) The student chooses the machine each day by rolling a fair six-sided dice. He chooses machine X if the dice roll is 2 or less, and machine Y otherwise. Write down the PDF describing the deposited thickness.

iv. (8 points) The student decides to alternate between machines X and Y. Starting with machine X on the first day, he proceeds to deposit the chemical product on the same plate for four days. Assuming that the deposited thicknesses on every day is independent, what is the probability that the total thickness deposited after four days is less than 4000 nm?

Solutions:

i. $P(950 < X < 1050) = P\left(\frac{950-1000}{\sqrt{625}} < Z < \frac{1050-1000}{\sqrt{625}}\right) = P(-2 < Z < 2)$

$= P(Z < 2) - P(Z < -2) = 0.9772 - 0.0228 = 0.954$

ii. $P(950 < Y < 1050) = P\left(\frac{950-980}{\sqrt{784}} < Z < \frac{1050-980}{\sqrt{784}}\right) = P(-1.071 < Z < 2.5) =$
$P(Z < 2.5) - P(Z < -1.071) = 0.9938 - 0.1421 = 0.852$

iii. $total\ probability: f_T(t) = \frac{1}{3} * \frac{1}{25\sqrt{2\pi}} e^{-\frac{(t-1000)^2}{2*625}} + \frac{2}{3} * \frac{1}{28\sqrt{2\pi}} e^{-\frac{(t-980)^2}{2*784}}$

iv. $thickness = X_1 + Y_2 + X_3 + Y_4$

$u = 1000 + 980 + 1000 + 980 = 3960$

$\sigma^2 = 625 + 784 + 625 + 784 = 2818$

$$P(T < 4000) = P\left(Z < \frac{4000 - 3960}{\sqrt{2818}}\right) = P(Z < 0.75) = 0.774$$

Qn 3.

The time to failure of a server can be modelled as an exponential distribution with a mean of 2 years. Answer the following questions, giving your answers to three decimal places.

i. (3 points) What is the probability that a server will fail within a year?
ii. (4 points) What is the probability that a server will fail between the first and second year?
iii. (8 points) Ten of these servers are networked together to form a distributed database (a process known as replication). What is the probability that at least 8 of these servers will still be functioning after a year?
iv. (5 points) There are 1000 of such servers in operation throughout the country. Estimate the probability that at least 620 of these servers will remain functioning after one year.

Solutions

$$\mu = 2 \rightarrow \lambda = 0.5, f(x) = \begin{cases} 0.5e^{-0.5x} & , x > 0 \\ 0 & , x \leq 0 \end{cases}$$

$$F(x) = P(X \leq x) = \begin{cases} 1 - e^{-0.5x} & , x > 0 \\ 0 & , x \leq 0 \end{cases}$$

i. $P(X \leq 1) = 1 - e^{-0.5} = 0.393$

ii. $P(X \leq 2) - P(X \leq 1) = 0.239$

iii. The probability that a server will still function after a year = 1 – 0.393 = 0.607

$_{10}C_8 * 0.607^8 * 0.393^2 + {}_{10}C_9 * 0.607^9 * 0.393^1 + 0.607^{10} = 0.179 = 17.9\%$

iv. X ~ Bin(1000, 0.607) ~ N(607, 238.65)

$$P(X \geq 620) = P\left(Z \geq \frac{620 - 0.5 - 607}{\sqrt{238.65}}\right) = P(Z \geq 0.81) = 1 - P(Z < 0.81)$$

$$= 1 - 0.7910 = 20.9\%$$

Qn 4.

The number of particles in a solution can be modelled as a Poisson distribution with an average concentration of 1.5 per ml. Answer the following questions, giving your answer to three decimal places.

i. (3 points) What is the probability that a 1 ml sample will contain less than 2 particles?

ii. (4 points) What is the probability that a 3 ml sample will contain more than 3 particles?

iii. (8 points) Samples of 3 ml are drawn one at a time until 5 samples are obtained with more than 3 particles. What is the average number of samples that have to be drawn?

iv. (5 points) A 3000 ml sample is drawn. Estimate the probability that there are at least 4400 particles in that sample?

Solutions

i. $P(X < 2) = P(X = 0) + P(X = 1) = e^{-1.5}(1 + 1.5) = 0.558$

ii. $P(X > 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3)$

$$= 1 - e^{-1.5*3}\left(1 + 4.5 + \frac{4.5^2}{2!} + \frac{4.5^3}{3!}\right) = 0.658$$

iii. If we define drawing a sample with more than 3 particles to be a "failure", this is a negative binomial distribution with p = 1 – 0.658 = 0.342 and r = 5.

The expected number of successes before hitting 5 failures, E[X] = 2.602

Average total number of samples = 5 + 2.6 = 7.6

iv. X ~N(4500, 4500)

$$P(X \geq 4400) = P\left(Z \geq \frac{4400 - 4500}{\sqrt{4500}}\right) = P(Z \geq -1.491) = 1 - P(Z < -1.491)$$

$$= 1 - 0.0680 = 0.932$$

Qn 5.

A binary message m, where m is equal either to 0 or to 1, is sent over an information channel. Assume that if $m = 0$, a voltage of $S = -1.5$ V is sent, and if $m = 1$, a voltage of $S = 1.5$ V is sent. The voltage received is X, where $X = S + E$, and E is normally distributed with $N(0, 0.62)$. If $X \leq 0.5$ V, then the receiver interprets it as $m = 0$, and if $X > 0.5$ V, then the receiver interprets it as $m = 1$. Answer the following questions, giving your answers to three significant figures.

i. (5 points) If the source message is $m = 0$, what is the probability of an error at the receiver, i.e., the receiver interprets it as $m = 1$? Round the answer to four decimal places.

ii. (5 points) If the true message is $m = 1$, what is the probability of an error, i.e., the receiver interprets it as $m = 0$?

iii. (5 points) The source emits binary 1 with probability 0.7 and binary 0 with probability 0.3. For a random bit received, what is the probability that it will be interpreted correctly?

iv. (5 points) Bits from the same source as (iii) are being interpreted one at a time. What is the probability that the first error in interpretation occurs at the 5th bit received?

Solutions:

i. $m = 0, X = -1.5 + E, X \sim N(-1.5, 0.62), P(error) = P(X > 0.5)$

$$P\left(Z \geq \frac{0.5 - (-1.5)}{\sqrt{0.62}}\right) = P(Z \geq 2.54) = 1 - P(Z < 2.54) = 1 - 0.9945 = 0.00554$$

Therefore, $, P(error) = 0.00554$

ii. $m = 1, X = 1.5 + E, X \sim N(1.5, 0.62), P(error) = P(X \leq 0.5)$

$$P\left(Z \leq \frac{0.5 - 1.5}{\sqrt{0.62}}\right) = P(Z \leq -1.270) = 1 - P(Z \leq 1.27) = 1 - 0.8980 = 0.102$$

Therefore, $, P(error) = 0.102$

iii. $P_{correct} = 0.9945 * 0.3 + 0.898 * 0.7 = 0.927$

iv. Geometric distribution:

P(k=5) = 0.927^4 * (1-0.927) = 0.0539

Data page

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998