Name: ______________________________ ID #: ______________________________

CENG 114 – Probability and Statistical Methods for Engineers

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Mid-term Exam 1

Winter 2017, Feb 2 2017

Instructions:

1. Write your name on the top right of this page.
2. Please answer **ALL** questions as far as possible.
3. You have a total of 80 mins exactly.
4. Write your answers in the space provided following the questions. If you need additional space, you may request for additional blank sheets of paper during the exam. If you use additional blank sheets, you should mark the top left of each sheet with your name, ID#, question number, and today's date, e.g., "John Smith, A1234567, Q1, Feb 2 2017".
5. Indicative maximum points for each sub-question are provided only as a reference. Actual grading points may vary.
6. Write neatly and legibly. Illegible answers will be considered as wrong.

Question 1 (20 points): ____________

Question 2 (20 points): ____________

Question 3 (20 points): ____________

Question 4 (20 points): ____________

Question 5 (20 points): ____________

This exam has a total of 7 pages, including the cover page.

Qn 1.

Let the universal set be the outcomes of a fair, 20-sided die. We define the following events:

A: The outcome is odd.

B: The outcome is a prime number. Note that 1 is not considered a prime number.

C: The outcome is a multiple of 3.

D: The outcome is a number 6 through 16, including both 6 and 16.

For each of the following scenarios, write out the result in set notation *and* calculate the corresponding probability.

i. (5 points) $A \cap C$
ii. (5 points) $B \cup C^C$
iii. (5 points) $A^C \cap D^C$
iv. (5 points) $A \cap B \cap D^C$

i. $A \cap C = \{3, 9, 15\},\ P = \frac{3}{20} = 0.15$
ii. $B \cup C^C = \{1, 2, 3, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20\},\ P = \frac{15}{20} = 0.75$
iii. $A^C \cap D^C = \{2, 4, 18, 20\},\ P = \frac{4}{20} = 0.2$
iv. $A \cap B \cap D^C = \{3, 5, 17, 19\},\ P = \frac{4}{20} = 0.2$

Qn 2.

Cybersecurity is one of the most critical concerns of our time. At ceng114multiverse.com, the administrator has stipulated that a password must:

- Be at least 8 characters but no more than 10 characters.
- Can be formed from both lower and upper case letters (A-Z, a-z), numbers (0-9) and four special symbols (@ & % #).

Unless otherwise stated, assume that repeated letters, symbols and numbers are allowed. Give all answers to three significant figures.

i. (6 points) How many possible passwords are there?

ii. (6 points) To create complex password that is still easy to remember, one way is to partition the password. You have decided to create a nine-character password that comprises three lower/upper case letters, followed by a special character, followed by five numbers. An example is "AbC&12345". How many possible passwords of this form are there?

iii. (6 points) You are still creating a password of the same form as in part (iii), but you have imposed an additional rule that no repeated letters or numbers are allowed. Note that upper and lower case letters are not considered the same letters, i.e., "A" and "a" are considered distinct. How many password combinations are there?

Solutions:

i. Number of characters = 26 + 26 + 10 + 4 = 66

Number of possible passwords = $66^8 + 66^9 + 66^{10} = 1.59 \times 10^{18}$

ii. There are 52 options for the first three letters, 4 options for the symbol, and 10 options for each of the five numbers.

Number of possible passwords = $52^3 \times 4 \times 10^5 = 5.62 \times 10^{10}$

iii. With the no-repetition restriction, we can now try to assign the letters from left to right. We have 52 options for the first letter, but only 51 for the second letter, etc.

Number of possible passwords = $52 \times 51 \times 50 \times 4 \times 10 \times 9 \times 8 \times 7 \times 6 = 1.60 \times 10^{10}$

Qn 3.

A drawer contains 12 red socks, 8 blue socks, and 11 green socks. Two socks are selected randomly one after the other without replacement.

i. (6 points) Given the first sock is red, what is the probability the second sock is blue?
ii. (7 points) What is the probability that the second sock chosen is green?
iii. (7 points) What is the probability of selecting a pair of the same color?

i. $\text{P(1st red)} = \frac{12}{31}$

$$\text{P(1st red} \cap \text{2nd blue)} = \frac{12}{31} \cdot \frac{8}{30}$$

$$\text{P(2nd blue|1st red)} = \frac{\text{P(1st red} \cap \text{2nd blue)}}{\text{P(1st red)}} = \frac{4}{15} = 0.2667$$

ii. Let A_1 = 1st sock is green, A_2 = 1st sock is not green, B = 2nd sock is green

$$\text{P(2nd green)} = P(A_1)P(B|A_1) + P(A_2)P(B|A_2)$$

$$P(B) = \frac{11}{31} \cdot \frac{10}{30} + \frac{20}{31} \cdot \frac{11}{30} = 0.3548$$

iii. $\text{P(pair)} = \text{P(1st red)P(2nd red|1st red)} + \text{P(1st blue)P(2nd blue|1st blue)} + \text{P(1st green)P(2nd green|1st green)}$

$$\text{P(pair)} = \frac{12}{31} \cdot \frac{11}{30} + \frac{8}{31} \cdot \frac{7}{30} + \frac{11}{31} \cdot \frac{10}{30} = 0.3204$$

Qn 4.

There are three boxes of engineering parts. Box 1 contains 100 items of which 10% are defective. Box 2 contains 200 items of which 30% are defective and box 3 contains 300 items of which 15% are defective. An item is picked by first randomly selecting a box (all three boxes have equal probability of being selected), and then randomly picking an item from the selected box.

i. (3 points) What is the probability that the selected item is defective?

ii. (6 points) What is the probability that a selected item is from Box 1 given that it is defective?

iii. (6 points) What is the probability that a selected item is from Box 2 given that it is not defective?

iv. (5 points) Instead of the two-step picking process, the contents of all boxes are dumped into one large box and randomized. What is the probability that an item picked is defective?

In all cases, assume that any items picked are replaced in the box it came from afterwards, i.e., you start picking from boxes with exactly the same initial contents each time.

Solution

i. $\mathrm{P(Box\ 1) = P(Box\ 2) = P(Box\ 3)} = \frac{1}{3}$
$\mathrm{P(Defective\ |Box\ 1)} = 0.1$
$\mathrm{P(Defective|Box\ 2)} = 0.3$
$\mathrm{P(Defective|Box\ 3)} = 0.15$

Total Probability Theorem:
$\mathrm{P(Defective)} = \big(\mathrm{P(Defective|Box1) * P(Box\ 1)}\big) + \big(\mathrm{P(Defective|Box2) * P(Box\ 2)}\big) + (\mathrm{P(Defective|Box\ 3) * P(Box\ 3)})$.

$$\mathrm{P(Defective)} = \left(0.1 * \frac{1}{3}\right) + \left(0.3 * \frac{1}{3}\right) + \left(0.15 * \frac{1}{3}\right) = 0.1833$$

ii. Bayes Rule:

$$\mathrm{P(Box\ 1|Defective)} = \frac{\mathrm{P(Defective|Box\ 1) * P(Box\ 1)}}{\mathrm{P(Defective)}}$$

$$P(\text{Box 1}|\text{Defective}) = \frac{0.1 * \frac{1}{3}}{0.1833}$$

$$P(\text{Box 1}|\text{Defective}) = 0.18$$

iii. $P(\text{Not defective}) = 1 - P(\text{Defective})$
$= 1 - 0.1833$
$= 0.8167$

$P(\text{Not defective}|\text{Box 2}) = 1 - P(\text{Defective}|\text{Box 2})$
$P(\text{Not defective}|\text{Box 2}) = 1 - 0.3 = 0.7$

Bayes Rule:

$$P(\text{Box 2}|\text{Not defective}) = \frac{P(\text{Not defective}|\text{Box 2}) * P(\text{Box 2})}{P(\text{Not defective})}$$

$$P(\text{Box 2}|\text{Not defective}) = \frac{0.7 * \frac{1}{3}}{0.8167} = 0.2857$$

iv. To find out the probability in this case we will first find out total number of non-defective items and total number of items.

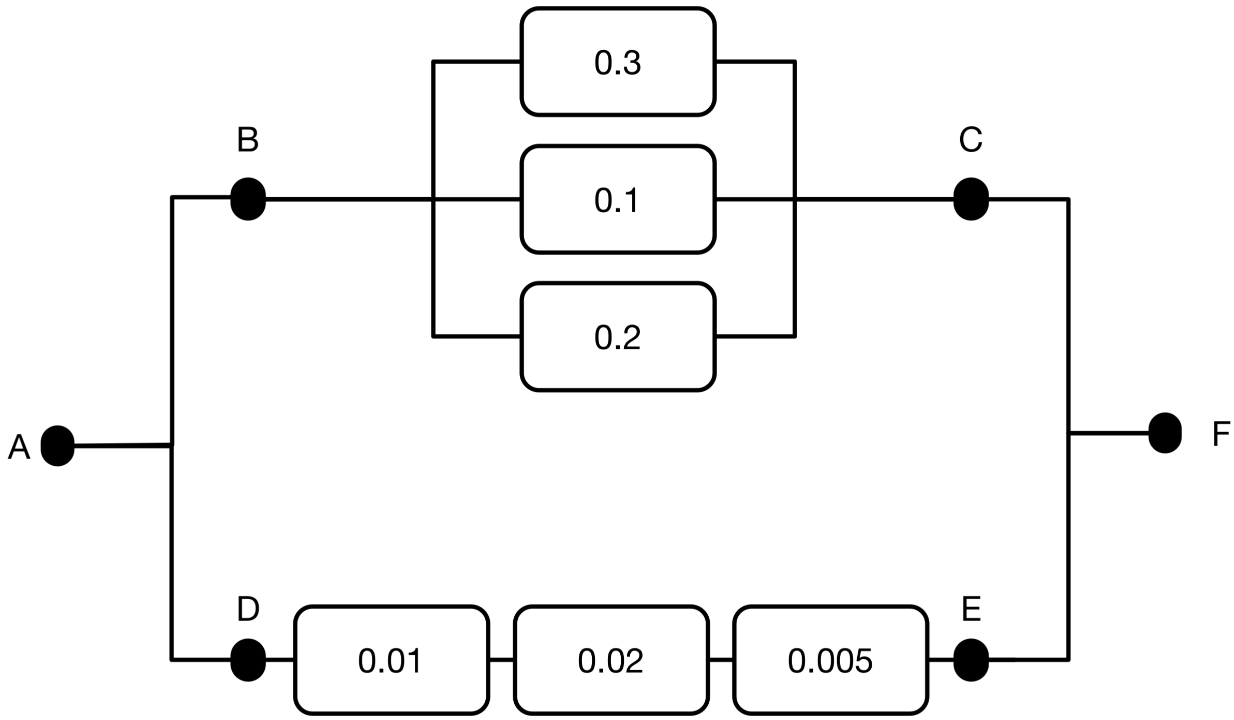
Total number of items in the box $= 100 + 200 + 300 = 600$
Total number of defective items in the box
$= (0.1 * 100) + (0.3 * 200) + (0.15 * 300) = 115$

$$P(\text{Picked item is defective}) = \frac{\text{Number of defective items in the box}}{\text{Total number of items in the box}}$$

$$= \frac{115}{600} = 0.1916.$$

Qn 5.

An IT message routing network is shown below. Each rounded rectangle is a repeater and the numbers indicate the *probability that a particular repeater fails*, i.e., a message cannot pass through. Answer the following questions (carry your results to four significant figures):



i. (6 points) What is the probability that a message can pass from B to C?
ii. (6 points) What is the probability that a message can pass from D to E?
iii. (4 points) What is the probability that a message can pass from A to F?
iv. (4 points) What is the probability that a message can pass from A to F, given that one of the repeaters between D and E fails?

Solutions

i. $P(B \to C) = 1 - 0.3 \times 0.1 \times 0.2$
$= 0.994$

ii. $P(D \to E) = (1-0.01) \times (1-0.02) \times (1-0.005)$
$= 0.9653$

iii. $P(A \to F) = 1 - (1-0.994) \times (1-0.9653)$
$= 0.9998$

iv. If one of the repeaters between D and E fails, the only way for a message to get from A to F is via B->C. So the conditional probability is simply equals P(B->C) = 0.994.