Challenge Set 3

Deadline: Feb 21 2017 at 5pm

Challenge problems are ***optional*** problems for those interested in testing their abilities. For each correct answer to a challenge question, bonus points of 0.1 are given towards the *final overall grade,* i.e., you can potentially earn up to 1.5 points towards the final grade if you get all questions correct. Proper workings must be shown to get any points, and there is no partial credit. Also, because these are bonus questions, instructors will not provide any help or hints (unlike typical problem or practice set questions where generous assistance will be provided) to be fair to all students. Please submit your solutions via the Turnitin assignment "Challenge Set 3" on TED@UCSD (you can simply take a good resolution photo/scan of your solutions with your student ID number and name clearly labelled and convert it to a PDF for upload) by the deadline.

Q1: Pat and Nat are dating, and all of their dates are scheduled to start at 9 pm. Nat always arrives promptly at 9 pm. Pat is highly disorganized and arrives at a time that is uniformly distributed between 8 pm and 10 pm. Let X be the time in hours between 8 pm and the time when Pat arrives. If Pat arrives before 9 pm, their date will last exactly 3 hours. If Pat arrives after 9 pm, their date will last for a time that is uniformly distributed between 0 and 3 - X hours. The date starts at the time they meet. Nat gets irritated when Pat is late and will end the relationship after the second date on which Pat is late by more than 45 minutes. All dates are independent of any other dates.

(a) What is the expected number of hours Nat waits for Pat to arrive?
(b) What is the expected duration of any particular date?
(c) What is the expected number of dates they will have before breaking up?

Solutions:

(a) Let A be the event that $X < 1$ hr, and Y be the number of hours that Nat has to wait for Pat to arrive.

$E[Y|A] = 0$ (because Pat arrives before Nat)

$E[Y|A^C] = 0.5$ (uniformly distributed between 9pm and 10pm)

$E[Y] = E[Y|A] * P(A) + E[Y|A^C] * P(A^C)$

$= 0.5 * 0.5 = 0.25$ hours

(b) Let Z be the expected duration of a date.

$$E[Z] = \int_0^2 E[Z \mid X] f_X(x)\,dx$$

$$= \int_0^1 3\frac{1}{2}dx + \int_1^2 (3-x)\frac{1}{2}dx$$

$$= 1.5 + 0.5[3x - \frac{x^2}{2}]_1^2$$

$$= 2.25$$

(c) P(X < 1:45) = 0.75

Let K be the number of successful dates to breakup (requiring 2 failed dates). This is a negative binomial distribution with $r = 2$, $p = 0.75$.

$$p_K(k) = \binom{k+1}{k} 0.75^k 0.25^2$$

$$E[K] = \frac{pr}{1-p} = \frac{0.75 \times 2}{1-0.75} = 6$$

So the number of dates they will have before breaking up is 6 + 2 (including the two failed dates) = 8.

Q2: Two machines are producing parts for car engines. The first machine can produce 120 parts per hour, but it makes an average of 1 mistake per hour. The second machine can produce 150 parts per hour, but it makes an average of 3 mistakes per hour. The two machines function independently.

a. What is the probability that during a two-hour period, six total mistakes are made?
b. The factory shuts down when 6 mistakes are made in a two-hour shift. About how many consecutive shifts can there be with a 5% chance of shutting down on the final shift?

Solution:

a.

For machine 1, $\lambda = 1\ mistake/hour$ and $\tau = 2\ hours$, so $\lambda\tau = 2\ mistakes$.

$$p_X(k_1) = \frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!} = \frac{e^{-2}(2)^k}{2!} = \frac{2^k}{e^2 * k!}$$

Similarly for machine 2, $\lambda = 3\ mistakes/hour$ and $\tau = 2\ hours$, so $\lambda\tau = 6\ mistakes$.

$$p_X(k_2) = \frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!} = \frac{e^{-6}(6)^k}{k!} = \frac{6^k}{e^6 * k!}$$

However, the total probability theorem cannot be used because the events are not independent (2 mistakes on machine 1 means 4 mistakes on machine 2). We have to add up the probability of each individual case.

k_1 = # mistakes by machine 1	$p_X(k_1)$	k_2 = # mistakes by machine 1	$p_X(k_2)$	$p_X(k_1 + k_2 = 6) = p_X(k_1) * p_X(k_2)$
0	0.13534	6	0.16062	0.02174
1	0.27067	5	0.16062	0.04348
2	0.27067	4	0.13385	0.03623
3	0.18045	3	0.08924	0.01610
4	0.09022	2	0.04462	0.00403
5	0.03609	1	0.01487	0.00054
6	0.01203	0	0.00248	0.00003

Therefore, the sum of the last column is $P(6\ total\ mistakes) = \ 0.12215$.

b.

Choose the geometric random variable since we want consecutive successful shifts and a failure on the last shift – p = 0.12215, (1 - p) = 0.87785, k = unknown, $p_X(k) = 0.05$

$$p_X(k) = \ (1-p)^{k-1} * p = 0.05$$

$$0.87785^{k-1} * 0.12215 = 0.05$$

$$0.87785^{n-1} = 0.40933$$

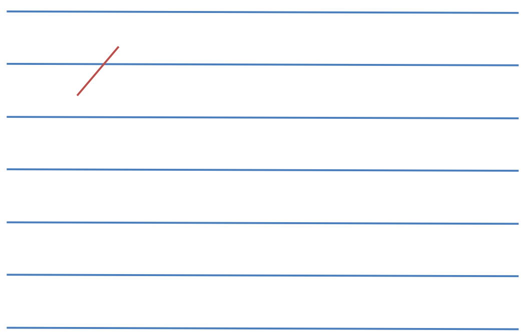
$$n - 1 = \frac{\log(0.40933)}{\log(0.87785)}$$

$$n = 7.86$$

There is a 5% chance that the factory will shut down on the 7th shift.

Q-3) A wooden rod of length 2a units is dropped at random onto a surface covered with infinitely long parallel metal rods (distance between the infinitely long parallel metal rods is also 2a units).Find the probability that wooden rod touches one of the metal rods.

Figure



Blue lines indicate infinitely long metal rods (spacing between the blue lines is 2a units) and orange line indicates the wooden rod.

Solution:

Let **α** denote the angle made by the wooden rod with respect to parallel lines (in counter clock wise direction).

Let **x** denote the distance from the wooden rod's center to the closest parallel rod.

Since we are dropping at random we can assume that **α** follows a uniform distribution (0, π) and **x** follows a uniform distribution (0, a). We can also assume that **x** and **α** independent random variables.

If $a * \sin(\boldsymbol{\alpha}) \geq \mathbf{x}$ **(**projection of the rod in direction perpendicular to the parallel lines**)** then the wooden rod will surely touch the metal rod.

So now our joint pdf is $f(\boldsymbol{x}, \boldsymbol{\alpha}) = \frac{1}{\pi a}$ we need to integrate the joint pdf in suitable region to get our required probability

Answer= $\iint_{[0,0]}^{[\pi, a\sin(\alpha)]} f(\boldsymbol{x}, \boldsymbol{\alpha}) d\,\boldsymbol{x}\, d\boldsymbol{\alpha} \quad = \frac{\mathbf{1}}{\boldsymbol{\pi a}} * \left(\int_{\mathbf{0}}^{\pi} a\sin(\boldsymbol{\alpha}) d(\boldsymbol{\alpha})\right) = \frac{\mathbf{2}}{\pi}$ **=0.636**