Solutions to CENG 114 Winter 2016 Mid-term Exam 2
Feb 18 2016

Qn 1.

Flaws along a magnetic tape follow a Poisson distribution with a mean of 0.2 flaw per meter. Let X denote the distance between two successive flaws.

i. (2 points) What is the mean of X?

ii. (4 points) What is the probability that there is more than one flaw in 10 meters of tape?

iii. (6 points) How many meters of tape need to be inspected so that the probability that at least one flaw is found is 90%?

iv. (8 points) A factory manufactures 50 km of tape per day. Estimate the probability that there are more than 9,900 flaws in any given day.

Ans:

i. $E[X] = \frac{1}{0.2} = 5$

ii. Let Y denote the number of flaws in 10 meters.

$$\begin{aligned} P(Y > 1) &= 1 - P(Y = 0) - P(Y = 1) \\ &= 1 - e^{-2}(1 + 2) \\ &= 0.594 \end{aligned}$$

iii. Let Z be the number of flaws in l meters of tape.

$$\begin{aligned} P(Z \geq 1) &= 1 - P(Z = 0) \\ &= 1 - e^{-0.2l} \\ &= 0.9 \\ \Rightarrow e^{-0.2l} &= 0.1 \\ -0.2l &= \ln(0.1) \\ l &= 11.51 \end{aligned}$$

iv. Let M be the number of flaws in 50 km of tape.

$$\begin{aligned} P(M > 9900) &= P\left(Z > \frac{9900 - 10000}{\sqrt{10000}}\right) \\ &= P(Z > -1) \\ &= P(Z < 1) \\ &= 0.8413 \end{aligned}$$

Qn 2.

A joint probability distribution function (PDF) is given by:

$$f_{X,Y}(x,y) = \begin{cases} xe^{-y} & 0 \le x \le c, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Answer the following questions:

i. (5 points) Calculate the value of *c* that makes this a valid PDF.
ii. (3 points) Calculate $P(X < 0.5, Y < 0.5)$.
iii. (2 points) Calculate $P(X < 0)$.
iv. (5 points) Derive the marginal PDF of X, $f_X(x)$.
v. (5 points) Calculate E[X].

Ans:

i.

$$\int_0^c \int_0^\infty xe^{-y}dydx = 1$$

$$\int_0^c x[-e^{-y}|_0^\infty]dx = 1$$

$$\left.\frac{x^2}{2}\right|_0^c = 1$$

$$c = \sqrt{2}$$

ii.

$$P(X < 0.5, Y < 0.5)$$

$$= \int_0^{0.5} \int_0^{0.5} xe^{-y}dydx$$

$$= [-e^{-y}|_0^{0.5}]\left.\frac{x^2}{2}\right|_0^{0.5} = (1 - e^{-0.5})\frac{0.5^2}{2}$$

$$= 0.0492$$

iii.

$$P(X < 0) = 0$$

iv.

$$f_X(x) = \int_0^\infty xe^{-y}dy$$
$$= \begin{cases} x & \text{if } 0 \le x \le \sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$

v.

$$E[X] = \int_0^{\sqrt{2}} xxdx$$
$$= \left.\frac{x^3}{3}\right|_0^{\sqrt{2}} = 0.9428$$

Qn 3.

Similar to radiocarbon dating, uranium-lead (U-Pb) dating is a method for dating rocks. In one variant of this method, the relevant radioactive decay process is from ^{238}U to ^{206}Pb, which can be modeled as an exponential random variable. The half-life of ^{238}U is 4.468×10^9 years.

Answer the following questions:

i. (5 points) What is the rate constant of the radioactive decay process in year^{-1}?

ii. (10 points) A sample of rock containing zircon ($ZrSiO_4$) is dated. Zircon incorporates U, but strongly rejects Pb, *i.e.*, any Pb found can be assumed to be from radioactive decay. The ratio of the atomic amount of ^{206}Pb to ^{238}U, $\frac{N_{Pb}}{N_U}$, in the sample is 0.4. Estimate the age of the rock. Hint: The original number of ^{238}U is equal to the sum of the number ^{206}Pb and ^{238}U today because each decayed ^{238}U forms exactly one ^{206}Pb, *i.e.*, $N_{Pb} + N_U$.

iii. (5 points) Estimate the ratio of ^{206}Pb to ^{238}U in a sample of rock aged 4 billion years old.

Ans:

i. The relationship between rate constant and half life time is:

$$1-e^{-\lambda t_{1/2}}=0.5;$$
$$\lambda=-\frac{\ln(0.5)}{t_{1/2}}=-\frac{\ln(0.5)}{4.468\times10^{9}}=1.551\times10^{-10}\text{ year}^{-1}$$

ii. The decay amount is:

$$\frac{N_{Pb}}{N_U}=0.4;\frac{N_{Pb}}{N_U+N_{Pb}}=\frac{1}{1/0.4+1}=0.28571$$

$$F(t)=1-e^{-\lambda t}=0.2857;\ -\lambda t=\ln(0.71429)$$

$$t=\frac{\ln(0.71429)}{-1.551\times10^{-10}}=2.169\times10^{9}\,years=2.169\text{ billion years}$$

iii. Given t = 4 billion years

$$F(t)=1-e^{-\lambda t}=1-e^{-1.551\times10^{-10}\times4\times10^{9}}=0.4623;$$

$$\frac{N_{Pb}}{N_{Pb}+N_U}=\frac{1}{1+\dfrac{N_U}{N_{Pb}}}=0.4623;$$

$$\frac{N_{Pb}}{N_U}=0.8597$$

Qn 4.

In an unspecified country, the total cholesterol level for adults can be modeled as a normally distributed random variable X with a mean of 159.2 mg/dl, and 84.13% of adults have a cholesterol level less than 200 mg/dl.

i. (2 points) Determine the standard deviation of the distribution of X.

ii. (4 points) What is the level of cholesterol that is exceeded by 90% of the population?

iii. (6 points) An adult is at moderate health risk if his cholesterol level is more than 0.5 standard deviations, but less than 2 standard deviations above the mean. What percentage of the population is at moderate health risk according to this criterion?

iv. (8 points) Five adults are selected independently and at random from the population. What is the probability that exactly three of these adults have cholesterol level less than 170 mg/dl?

Ans:

i. Given that 84.1% of adults have a cholesterol level less than 200 mg/dl, z-score = 1.00 from given z-table.

$$z=\frac{x-\mu}{\sigma}=\frac{200-159.2}{\sigma}=1$$

$$\sigma=40.8$$

ii. z-score = -1.28 is where the level of cholesterol exceeded by 90% of the population

$$z = \frac{x-\mu}{\sigma} = \frac{x-159.2}{40.8} = -1.28$$
$$x = 106.98$$

iii. His cholesterol level can be expressed as:

$$\mu + 0.5\sigma < x < \mu + 2\sigma;$$
$$\frac{\mu + 0.5\sigma - \mu}{\sigma} < z < \frac{\mu + 2\sigma - \mu}{\sigma}$$
$$0.5 < z < 2$$
$$0.6915 < \text{area under the curve} < 0.9773$$

Area within the range is 0.2858

Therefore, 28.58% of the population is at moderate risk according to this criterion.

iv. The probability of an adult has cholesterol level less than 170 mg/dl is:

$$P(x < 170),\ z = \frac{170-159.2}{40.8} = 0.26$$
$$P(z < 0.26) = 0.6026$$

The number of adults with X < 170 out of 5 can be modeled as B(5, 0.6026). Therefore, the requested probability is:

$$P = {}^5C_3 P(x<170)^3 P(x \geq 170)^2 = 10(0.6026)^3(1-0.6026)^2 = 0.3456$$

Qn 5.

Three machines X, Y and Z are used to produce an electrolyte for rechargeable Li-ion batteries. The molar concentration of Li^+ in the electrolyte produced from all three machines can be modeled as normal random variables with standard deviation of 0.1 M. Due to calibration differences, the mean concentration of Li^+ produced in electrolyte from machines X, Y, and Z are 0.88 M, 0.95 M and 1.1 M respectively.

i. (6 points) A satisfactory electrolyte must have a Li^+ concentration of at least 1 M. What are the probabilities of machines X, Y and Z producing a satisfactory electrolyte? Note that three probabilities need to be given.

ii. (12 points) A batch of 1,000 electrolyte samples are sent to a customer, of which 350 are from machine X, 450 are from machine Y, and 200 are from machine Z. A sample is chosen at random and is tested to be satisfactory. What is the probability that the tested sample came from machine Z?

iii. (2 points) The instrument for testing the electrolyte has an error E that is normally distributed with mean 0 and standard deviation of 0.1 times the molar concentration of Li^+ of the sample. If Y denotes the actual molar concentration from machine Y, write down the joint PDF of Y and E. You just need to write down the correct form and do not need to simplify your answer.

Ans:

i. First, we need to calculate the probability of each machine which can produce concentration of at least 1M.

$P(X \le 1)$, $z = \dfrac{1-0.88M}{0.1M} = 1.2$, with area = 0.8849

$P(Y \le 1)$, $z = \dfrac{1-0.95M}{0.1M} = 0.5$, with area = 0.6915

$P(Z \le 1)$, $z = \dfrac{1-1.1M}{0.1M} = -1$ with area = 0.1586

Then the request probabilities are:

$P(X > 1) = 1 - 0.8849 = 0.1151$

$P(Y > 1) = 1 - 0.6915 = 0.3085$

$P(Z > 1) = 1 - 0.1586 = 0.8414$

ii. S: satisfactory sample; Z: sample from Z machine

$$P\left(Z|S\right)=\frac{P\left(S|Z\right)P\left(Z\right)}{P\left(S|X\right)P\left(X\right)+P\left(S|Y\right)P\left(Y\right)+P\left(S|Z\right)P\left(Z\right)}$$
$$=\frac{0.8414\times 0.2}{0.1151\times 0.35+0.3085\times 0.45+0.8414\times 0.2}$$
$$=0.4844$$

iii. PDF of the normal distributed variable can be expressed as:

$$f_{YE}\left(y,e\right)=f_{Y}\left(y\right)f_{E|Y}\left(e|y\right)$$
$$=\frac{1}{0.1\sqrt{2\pi}}e^{\frac{-\left(y-0.95\right)^2}{2\left(0.1\right)^2}}\frac{1}{\left(0.1y\right)\sqrt{2\pi}}e^{\frac{-\left(e\right)^2}{2\left(0.1y\right)^2}}$$