Name: ______________________________

CENG 114 – Probability and Statistical Methods for Engineers

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Mid-term Exam 1

Winter 2016, Jan 28 2016

Instructions:

1. Write your name on the top right of this page.
2. Please answer **ALL** questions as far as possible.
3. You have a total of 80 mins exactly.
4. Write your answers in the space provided following the questions. If you need additional space, you may request for additional blank sheets of paper during the exam. If you use additional blank sheets, you should mark the top left of each sheet with your name and the question number and today's date, e.g., "John Smith, Q1, Jan 28 2016".
5. Indicative maximum points for each sub-question are provided only as a reference. Actual grading points may vary.
6. Write neatly and legibly. Illegible answers will be considered as wrong.

Question 1 (20 points): ____________

Question 2 (20 points): ____________

Question 3 (20 points): ____________

Question 4 (20 points): ____________

Question 5 (20 points): ____________

This exam has a total of 8 pages, including the cover page.

Qn 1.

A lot of 100 semiconductor chips contains 20 that are defective. Three are selected randomly, **without replacement**, from the lot.

(a) What is the probability that the second one selected is defective? (10 pts)

(b) What is the probability that the second and third selected ones are defective given that the first one was defective? (5 pts)

(c) How does the answer to part (b) change if chips selected were replaced prior to the next selection? (5pts)

Answers:

(a) Let A_i denote the event that a non-defective chip is obtained on the i-th selection

$$P(A_2^C) = P(A_1^C)P(A_2^C | A_1^C) + P(A_1)P(A_2^C | A_1)$$
$$= \frac{20}{100} \times \frac{19}{99} + \frac{80}{100} \times \frac{20}{99}$$
$$= 0.2$$

(b) Given that first one was defective, the probability of second and third selected ones are defective:

$$P = P(\text{second and third are } defective \,|\, first\ is\ defective)$$
$$= \frac{19}{99} \times \frac{18}{98} \approx 0.035$$

(c)

$$P = P(second\ and\ third\ are\ defective,\ if\ replaceable)$$
$$= \left(\frac{20}{100}\right)^2 \approx 0.04$$

Qn 2.

There is a deck of 21 cards numbered 1 through 21 and a fair 6-sided die. We define the following events:

A: Prime numbered cards are selected.

B: Even numbered cards are selected.

C: Sample space of the 6-sided die.

a. For each of the events below, write down all elements in the form {1, 2, 3, …} **and** calculate the probability of that event. (6 points)
   i. $A \cap B^C$
   ii. $A \cup C$
   iii. $A^C \cap B \cup C$

b. If you draw 2 cards from the deck with replacement, what is the probability that the sum of the values of the 2 cards you draw will be an even number? (8 points)

c. If you draw 5 cards from the deck without replacement, what is the probability that your hand contains 10, 11, and 12? (6 points)

Ans:

a.

A = {2, 3, 5, 7, 11, 13, 17, 19}

B = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20}

C = {1, 2, 3, 4, 5, 6}

i. $A \cap B^C$ = {3, 5, 7, 11, 13, 17, 19}

ii. $A \cup C$ = {1, 2, 3, 4, 5, 6, 7, 11, 13, 17, 19}

iii. $A^C \cap B \cup C$ = {1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 16, 18, 20}

b. There are two ways of getting the sum of 2 cards even: both cards odd or both cards even. Since the cards are drawn with replacement, the first and the second draw are independent, therefore:

P(sum of two cards is even) = P(two odd cards) + P(two even cards)

= 11/21 * 11/21 + 10/21 * 10/21

= 121/441 + 100/441 = 221/441 = 0.501

c. P(choose 5 cards containing 10, 11, and 12) = (1*1*1* $^{18}C_2$)/ $^{21}C_5$

= 153/20349 = 0.0075

Qn 3.

You are building a datacenter consisting of servers. The probability of a server failure in any given day is 0.05. Servers fail independently from each other.

(a) What is the probability that a server works with no failure for 5 consecutive days? (5 pts)

(b) If the datacenter has 3 servers, what is the probability that at least one server works without failure for 5 consecutive days. (5 pts)

(c) If we want to make sure the probability of at least one server works without failure for 7 consecutive days is higher than 99.9%, what is the minimum number of servers the data center need to have? (10 pts)

Solution:

(a) Probability of server works for 5 consecutive days is, let P(S) be the probability of a server works in a day, P(S) = 0.95:

$$P(A) = P(S)^5 = 0.95^5 = 0.774$$

(b) And the complement event of at least one server works without failure for 5 consecutive days is all three servers fails at least once during the 5 consecutive days.

$$P(C) = 1 - P(A^C)^3 = 1 - (1 - 0.774)^3 = 1 - 0.0115 = 0.9885$$

(c) The probability that a server has at least one failure in 7 consecutive days is:

$$P(D) = 1 - P(S)^7 = 1 - 0.95^7 = 0.302$$

If there are x servers, the probability that at least one server works without failure in 7 days is:

$$P(> 1\ server\ work\ for\ 7\ days\ without\ failure)$$
$$= 1 - P(0\ servers\ work\ without\ failure\ in\ 7\ days) = 1 - P(D)^x$$

We want the probability of at least one server works without failure for 7 consecutive days higher than 99.9%.

$$1 - P(D)^x > 0.999$$
$$0.302^x < 0.001$$
$$x log(0.302) < \log(0.001)$$

Solving the above problem, x > 5.76. So there needs to be a least 6 servers.

Qn 4.

An office uses a certain type of printer. The probabilities of poor print quality if there is no printer problem, misaligned paper, high ink viscosity, or printer-head debris are 0, 0.3, 0.4, and 0.6, respectively. The probabilities of no printer problem, misaligned paper, high ink viscosity, or printer-head debris are 0.8, 0.02, 0.08, and 0.1, respectively.

a. (10 points) Determine the probability of high ink viscosity given poor print quality.

b. (10 points) If the printer has good print quality, what is the probability of there being no printer problem?

Ans:

Let N, M, V, D denote the events that there is no printer problem, misaligned paper, high ink viscosity, and printer-head debris respectively. Let P denote the event that there is poor print quality.

$P(N) = 0.8$

$P(M) = 0.02$

$P(V) = 0.08$

$P(D) = 0.1$

$P(P|N) = 0$

$P(P|M) = 0.3$

$P(P|V) = 0.4$

$P(P|D) = 0.6$

For a, we want

$P(V|P) = P(P|V) * P(V) / P(P)$

$= P(P|V) * P(V) / [P(P|V) * P(V) + P(P|D) * P(D) + P(P|N) * P(N) + P(P|M) * P(M)]$

$= 0.4 * 0.08 / [0.4 * 0.08 + 0.6 * 0.1 + 0 * 0.8 + 0.3 * 0.02]$

$= 0.327$

b.

We want $P(N|P^C) = P(P^C|N) * P(N) / P(P^C)$

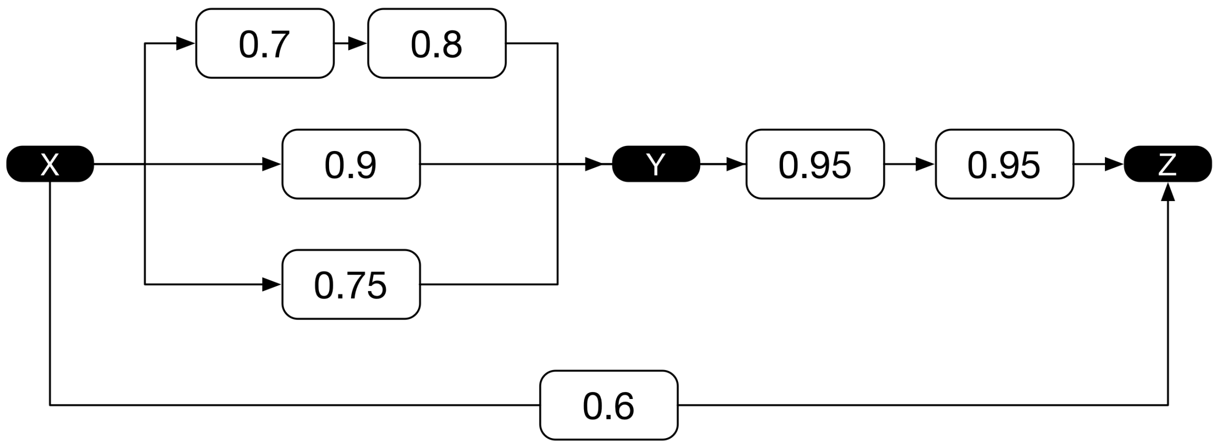
$= P(P^C|N) * P(N) / [P(P^C|N) * P(N) + P(P^C|V) * P(V) + P(P^C|D) * P(D) + P(P^C|M) * P(M)]$

$= 1 * 0.8 / (1 * 0.8 + (1 – 0.4) * 0.08 + (1 – 0.6) * 0.1 + (1 – 0.3) * 0.02)$

$= 0.887$

Qn 5.

A local rail network that connects three cities X, Y and Z is given below.



Each box with a white background shows the probability of that particular part of the rail network being operational. Answer the following questions:

i. (6 points) What is the probability that there is an operational path between X and Y?
ii. (6 points) What is the probability that there is an operational path between Y and Z?
iii. (8 points) What is the probability that there is an operational path between X and Z?

Solution:

i.

P(X->Y) = 1 – P(no path from X->Y)

= 1 – (1-0.9) * (1-0.75) * (1 – 0.7 * 0.8)

= 0.989

ii.

P(Y -> Z) = 0.95 * 0.95 = 0.9025

iii.

P(X->Z path 1) = P(X->Y) * P(Y->Z) = 0.989 * 0.9025 = 0.8926

P(X->Z) = 1-P(no path X->Z)

= 1 – (1-0.8926) * (1-0.6) = 0.957