CENG/NANO 114

Probability and Statistical Methods for Engineers

Problem Set 3

Note: Questions denoted with a * are meant to be more difficult than usual. Hints are provided for a selected number of questions.

1. Statistics show that an electronic component has life expectancy (in days) follow the following probability density function:

$$p(x) = \begin{cases} \frac{1}{x^2} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

i. Probability component lasts between 0 and 10 days?
ii. Probability it lasts more than 10 days?
iii. Find the CDF for the life expectancy.

2. The life of a semiconductor laser at a constant power is normally distributed with a mean of 7000 hours and a standard deviation of 600 hours.

i. What is the probability that a laser fails before 5000 hours?
ii. What is the life in hours that 95% of the lasers exceed?
iii. If three lasers are used in a product and they are assumed to fail independently, what is the probability that all three are still operating after 7000 hours?

3. A software has the number of failures N(t) over the time interval [0, t), which can be described by a homogeneous Poisson process $\{N(t), t \geq 0\}$. A failure occurs every 4 hours on average.

i. What is the probability of at most 1 failure in [0, 8), at least 2 failures in [8, 16), and at most 1 failure in [16, 24) (time unit: hour)?
ii. What is the probability that the third failure occurs after 8 hours?

Hint: The rate of the process is $0.25[h^{-1}]$.

4. Assume that in a digital communication channel, the number of bits received in error can be modeled by a binomial random variable, and assume that the probability that a bit is received in error is 10^{-5}. If 16 million bits are transmitted, estimate the probability that 150 or fewer errors occur? (Hint: You need to choose the right approximation)

5. The monthly demand for MMR vaccine is approximately normally distributed with a mean and standard deviation of 1.2 and 0.4 million doses, respectively. Suppose that the demands for different months are independent, and let Z denote the demand for a year (in millions of does). Determine the following:

i. Mean, variance, and distribution of Z

ii. $P(Z < 14.4)$

iii. $P(12<Z<17)$

iv. Value for c such that $P(Z < c) = 0.99$

6. The photoresist thickness in semiconductor manufacturing has a mean of 15 micrometers and a standard deviation of 1 micrometer. Assume that the thickness is normally distributed and that the thicknesses of different wafers are independent.

i. Determine the probability that the average thickness of 15 wafers is either greater than 16 or less than 14 micrometers.

ii. Determine the number of wafers that need to be measured such that the probability that the average thickness exceeds 16 micrometers is 0.01.

iii. If the mean thickness is 10 micrometers, what should the standard deviation of thickness equal so that the probability that the average thickness of 15 wafers is either more than 16 or less than 14 micrometers is 0.001?

7. The systolic and diastolic blood pressure values (mm Hg) are the pressures when the heart muscle contracts and relaxes (denoted as Y and X, respectively). Over a collection of individuals, the distribution of diastolic pressure (X) is normal with mean 75 and standard deviation 10. The systolic pressure is conditionally normally distributed with mean 1.5 x when $X = x$ and standard deviation of 12. Determine the following:

i. Conditional probability density function $f_{Y|75}(y)$ of Y given $X = 75$

ii. $P(Y<115|X=75)$

iii. $E(Y|X=75)$

iv. Determine the joint PDF $f_{XY}(x,y)$ and identify the mean and variance of Y and the correlation between X and Y

8. Let the random variable X denote the time until a computer server connects to your machine (in milliseconds), and let Y denote the time until the server authorizes you as a valid user (in milliseconds). Each of these random variables measures the wait from a common starting time and $X < Y$. Assume that the joint probability density function for X and Y is

$$f_{XY}(x,y) = 6\times10^{-6}\exp(-0.001x - 0.002y) \quad \text{for X<Y.}$$

Determine the conditional probability density function for Y given that $X = x$.
(Hint: Get the marginal density function for X first and then use the conditional probability equation to calculate the conditional PDF for Y given that $X=x$)

9. The distance between major cracks in a highway follows an exponential distribution with a mean of six miles.

i. What is the probability that there are no major cracks in a 12-mile stretch of the highway?
ii. What is the probability that there are two major cracks in a 12-mile stretch of the highway?
iii. What is the standard deviation of the distance between major cracks?
iv. What is the probability that the first major crack occurs between 15 and 18 miles of the start of inspection?
v. What is the probability that there are no major cracks in two separate six-mile stretches of the highway?
vi. Given that there are no cracks in the first six miles inspected, what is the probability that there are no major cracks in the next 12 miles inspected?

10.* In quantum mechanics, the wavefunction of the hydrogen atom for quantum numbers n = 1, l = 0 and m = 0 in <u>3D polar coordinates</u> is given by

$$\psi(r,\theta,\phi) = ke^{-\frac{r}{a_0}} \text{ where } a_0 \text{ is the Bohr radius and } k \text{ is a constant.}$$

The square of the wavefunction is also the probability density function of finding an electron at a particular position (r, θ, ϕ) with r taking values from 0 to +∞. Answer the following questions:

i. Derive the value of k in terms of the Bohr radius using the axioms of probability.
ii. Write down the radial PDF (i.e., no angular components)?
iii. Calculate the value of r that results in the maximum probability of the <u>radial PDF</u> at a distance r.
iv. Calculate the expectation of the radial PDF.

Hint: You have to integrate by parts in spherical coordinates. It may seem to require extremely arduous mathematical manipulations, but there is a shortcut to do the integration once you realize the consequences of the limits.