Nano / Ceng 114 Solution for Problem Set 2

1 i) Using normalization axiom:

$$\sum_{x=1}^{5} P_X(x) = 1$$

Given by the problem $P_X(x) = \frac{k}{x}$ (k is a constant)

Thus, we get: $\sum_{x=1}^{5} P_X(X=x) = \frac{k}{1} + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} + \frac{k}{5} = 1$

$$\frac{137}{60} k = 1 \Rightarrow k = \frac{60}{137}$$

ii) mean: $E(X) = \sum_{x=1}^{5} x P_X(x) = \sum_{x=1}^{5} x \cdot \frac{k}{x} = 5k = \frac{300}{137} = 2.19$

Variance: $Var(X) = E[X^2] - E[X]^2$

$$= \sum_{x=1}^{5} x^2 \cdot P_X(x) - (5k)^2 = \sum_{x=1}^{5} x^2 \cdot \frac{k}{x} - (5k)^2$$

$$= k + 2k + 3k + 4k + 5k - (5k)^2$$

$$= 15k - (5k)^2 = 1.77$$

(You can also get the right answer by using the definition of variance. But the calculation may be a bit more complicated.)

2 i) Marginal PMF $P_X(x) = \sum_{y} P_{X,Y}(x,y)$; $P_Y(y) = \sum_{x} P_{X,Y}(x,y)$

From the table of this problem, we can get $P_{X,Y}$ for each possible X, Y.

So $P_X(X=1) = P_{1,1} + P_{1,2} + P_{1,3} = 0 + 0.2 + 0.02 = 0.22$ for $X=1$

Similarly, we get $P_X(X=2) = P_{2,1} + P_{2,2} + P_{2,3} = 0.05 + 0.1 + 0.3 = 0.45$

$P_X(X=3) = P_{3,1} + P_{3,2} + P_{3,3} = 0.3 + 0 + 0.03 = 0.33$

The same for P_Y (marginal PMF for Y)

$P_Y(Y=1) = P_{1,1} + P_{2,1} + P_{3,1} = 0 + 0.05 + 0.3 = 0.35$

$P_Y(Y=2) = P_{1,2} + P_{2,2} + P_{3,2} = 0.2 + 0.1 + 0 = 0.3$

$P_Y(Y=3) = P_{1,3} + P_{2,3} + P_{3,3} = 0.02 + 0.3 + 0.03 = 0.35$

8 iv) $E(Y) = n \cdot p$ Y: the number of passing the test

$= 100 \times 0.75$ $p = 0.75$, n: number of tests.

$= 75$

$Var(Y) = np(1-p) = 100 \times 0.75 \times 0.25 = 18.75$

$\sigma = \sqrt{Var(Y)} = \sqrt{18.75} = 4.33$

9 i) a mean of 2 cracks per mile ⇒ for 6 miles of highway, the mean will be $2 \times 6 = 12$ cracks ($\lambda = 12$)

$$P_X(k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad (k = 0, 1, 2 \cdots)$$

$$P_X(k=0) = e^{-12} \frac{12^0}{0!} = e^{-12} = 6.14 \times 10^{-6}$$

ii) For 1.5 miles of highway, the mean will be $2 \times 1.5 = 3$ ($\lambda = 3$)

$$P_X(k \geq 1) = 1 - P_X(k=0) = 1 - e^{-3} \frac{3^0}{0!} = 0.95$$

iii) The Poisson distribution won't agree with the problem any more, because the Poisson distribution is a good approximation for $B(n,p)$ when $np = \lambda$ and n is large & p is small. If the number of cracks is related to vehicle load and some sections have a heavy load while others carry a light load, the n&p in different sections are not the same. In some sections, n may be small, so we can't use the Poisson approximation in such condition.

10 X: steps forwards   Y: steps backwards.

Assume n steps have been made

$E(X) = n \cdot p$

$E(Y) = n(1-p)$

$E(\text{displacement}) = dE(X-Y) = d[np - n(1-p)] = dn(2p-1)$

when $p = 0.5$, $E(\text{displacement}) = 0$