CENG/NANO 114

Probability and Statistical Methods for Engineers

Problem Set 1

1. Consider rolling a twenty-sided icosahedron die. The universal set Ω contains all the possible outcomes of a roll. We define the following sets:
   a. A = {x | x is odd}
   b. B = {x | x≤10}
   c. C = {x | x is even}
   d. D = {x | x is a prime number}

   Write down the results of the following set operations, explicitly listing all elements (e.g., {1, 2, 3, 4}).

   i. $A \cap B$
   ii. $(A \cap B)^c$
   iii. $A \cap C$
   iv. $A \cup B$
   v. $B^c \cup D$
   vi. $A \cap B \cap D$

2. An experimentalist is studying the effects of temperature, pressure and type of catalyst on yield from a certain chemical reaction. Three different temperatures, four different pressures, and five different catalysts are under consideration.
   i. If a particular experiment involves the use of a single temperature, pressure, and catalyst, how many possible sets of the three experimental conditions are there?
   ii. How many possible experimental runs are there that involve use of the lowest temperature and one of the two lowest pressures?

3. There are 9 visitors and 3 tour guides in a group. They want to cross a river by the three boats at the riverside. The shapes of these boats are different, so they can contain different number of people. The small, medium and large boats can contain 3, 4 and 5 persons respectively. The visitors and guides are randomly divided into the three boats. What is the probability that every boat contains a tour guide?

4. A rental car service facility has 10 foreign cars and 15 domestic cars waiting to be serviced on a particular Sunday. Because there are so few mechanics working that day, only 6 cars can be serviced. If the 6 cars are chosen at random, what is the probability that *more than* 3 foreign cars are selected?

5. There is a product that can be produced by A company and B company. The defective

probability of this product is 1% for A company and 2% for B. Now we have a batch of products 60% from A and 40% from B. A sample was randomly selected from this batch and tested, and was found to be defective. What is the probability of this sample is produced by A company?

6. There are 4 types of molecules to form a chain. Three molecules of type A, three molecules of type B, three molecules of type C and three of D are used in this chain reaction. One example chain molecule can be *ABCDABCDABCD*.
   a. How many such chain molecules are there? (Hint: consider three molecules in the same type are distinguishable (A_1,A_2,A_3,......), then how is this number reduced when see three molecules in the same type such as As the same?)
   b. Suppose a chain molecule of the above description is randomly selected. What is the probability that all three molecules of each type end up next to one another (such as in *AAABBBCCCDDD*)?

7. Only 1 in 1000 adults is afflicted with a rare disease for which a diagnostic test has been developed. The test is such that, when an individual actually has the disease, a positive result will occur 99% of the time, while an individual without the disease will show a positive test result only 2% of the time. If a randomly selected individual is tested and the result shows positive.
   a. What is the probability that the individual has the disease?
   b. Normally, the doctor will not diagnose the individual has such disease when he or she's first test result is positive. The individual will be asked to do the test again. What is the probability that the individual has the disease if he or she gets both positive results in two diagnostic tests? (Hint: You can use tree diagram to help with analysis and solve via Bayes' rule.)

8. A quality inspector tests a product by selecting k random samples out of each batch of 200 items. When any item in the test sample is tested to be defective, that batch fails. Otherwise, the batch passes. What is the minimum required value of k if the quality inspector wishes to make sure that the probability of erroneously passing a batch (known as a false negative) with six defective items is less 10%?

9. In the lectures, we discussed the number of ways in which a password can be created out of the alphanumeric characters (A-Z, a-z and 0-9). For even better security, we can require that a password contains at least one special character from these six:

   {#, $, %, *, &, @}

   How many possible passwords with six characters are there with at least one special character if:
   a. Repetition of any character is not allowed?
   b. Repetition is allowed?

10. A credit card contains 16 digits. It also contains the month and year of expiration. Suppose there are 1 million credit card holders with unique card numbers. A hacker

randomly selects a 16-digit credit card number.

a. What is the probability that it belongs to a user when only consider the 16 digits?
b. Suppose a hacker has a 20% chance of correctly guessing the year your card expires and randomly selects 1 of the 12 months. What is the probability that the hacker correctly selects the month and year of expiration?

11. Magnesium alkyls are used as homogenous catalysts in the production of linear low-density polyethylene (LLDPE), which requires a finer magnesium powder to sustain a reaction. Redox reaction experiments using four different amounts of magnesium powder are performed. Each experiment may or may not be further reduced in a second step using three different magnesium powder amounts. Each of these results may or may not be further reduced in a third step using three different amounts of magnesium powder.
    a. How many experiments are possible?
    b. If all outcomes are equally likely, what is the probability that the best result is obtained from an experiment that uses all three steps?
    c. Does the result in part (b) change if 5 or 6 or 7 different amounts are used in the first step? Explain.

12. Redundant array of inexpensive disks (RAID) is a technology that uses multiple hard drives to increase the speed of data transfer and provide redundancy. Suppose that the probability of any hard drive failing in a day is 0.0008 and the drive failures are independent.
    a. A RAID 0 scheme uses two hard drives, each containing a mirror image of the other. What is the probability of data loss? Assume that data loss occurs if both drives fail within the same day.
    b. A RAID 1 scheme splits the data over two hard drives. What is the probability of data loss? Assume that data loss occurs if at least one drive fails within the same day.

13. A source transmits a message (a string of symbols) through a noisy communication channel. Each symbol is 0 or 1 with probability p and $(1-p)$, respectively, and is received incorrectly with probability ε_0 and ε_1, respectively. So the probability of symbol 0 is received correctly is $(1-\varepsilon_0)$ and correct received probability for symbol 1 is $(1-\varepsilon_1)$. Errors in different symbol transmissions are independent.
    a. What is the probability of a symbol is received correctly?
    b. What is the probability that the string of symbols 0110 is received correctly?
    c. In an effort to improve reliability, each symbol is transmitted three times from the original symbol and the received string is decoded by the majority rule. For example, a 1 is transmitted as 111 and it is decoded at the receiver as a 1 if and only if the received three-symbol string contains at least two 1 s. What is the probability that a 1 is correctly decoded?
    d. Suppose that the scheme of part (c) is used. What is the probability that a symbol was 0 given that the received string is 011?

14. In statistical mechanics, the Einstein model is a model of a solid based on independent 3D

quantum harmonic oscillators. The energy of a quantum harmonic oscillator are quantized in levels with the following formula:

$$E_n = \bar{h}\omega\left(n + \frac{1}{2}\right) \text{ with n=0,1,2,3,4,...}$$

The probability of the oscillator being in a specific energy state is given by:

$$P(E_n) = \frac{e^{-\beta E_n}}{Z}$$

where Z is known as the partition function. You may treat $\bar{h}, \omega$ and β as parameters. Derive an expression for the partition function using the axioms of probability, simplifying your answer to a function of just $\bar{h}, \omega$ and β. Show all steps in the derivation.