CENG/NANO 114
Winter 2015
Solution for Problem Set 1

1.

Ω = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}
A = {1, 3, 5, 7, 9, 11, 13, 15, 17, 19}
B = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
C = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20}
D = {2, 3, 5, 7, 11, 13, 17, 19}

You need to understand the definitions of *Union* $\cup$, *Intersection* $\cap$, *and complement* ^{c}.

1) $A \cap B$ = {1, 3, 5, 7, 9}
2) $(A \cap B)^c$ = {2, 4, 6, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}
3) $A \cap C$ = {no element}= $\varnothing$
4) $A \cup B$ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19}
5) $B^c \cup D$ = {2, 3, 5, 7, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}
6) $A \cap B \cap D$ = {3, 5, 7}

2.

The temperature, pressure and type of catalyst are three independents factors.

1) For 3 temperature, 4 pressures and 5 types of catalyst,
   $3 \times 4 \times 5 = 60$
2) For 1 temperature, 2 pressures, 5 types of catalyst, similarly,
   $1 \times 2 \times 5 = 10$

3.

1) All possible arrangements, using partition formula:
   If we want to arrange $9+3=12$ people into 3 boats,
   each boat contains 3,4,5people,
   Thus, the total number of arrangement is
   $$\frac{12!}{3!4!5!} = 27720$$
2) To make sure each boat contains a tour guide, we firstly arrange the tour guides: $3! = 6$. Then arrange the visitors (2, 3, 4 seats remaining in each boat): $\frac{9!}{2!3!4!} = 1260$.
   Therefore, the probability is, $P = \frac{3! \times 1260}{27720} = \frac{3}{11}$

4.

1) All possible arrangements: ${}^{25}C_6 = 177100$
2) more than 3 foreign cars: 4, 5, 6 foreign cars

$$^{10}C_4{}^{15}C_2 + {}^{10}C_5{}^{15}C_1 + {}^{10}C_6{}^{15}C_0 = 22050 + 3780 + 210 = 26040$$

Therefore, we can get the probability:

$$P = \frac{^{10}C_4{}^{15}C_2 + {}^{10}C_5{}^{15}C_1 + {}^{10}C_6{}^{15}C_0}{^{25}C_6} = \frac{186}{1265} \quad \text{(Or 0.147)}$$

5.

Conditional probability, given that the sample is defective.

$$P(fromA \cap defective) = 0.6 \times 0.01 = 0.006$$

$$P(fromB \cap defective) = 0.4 \times 0.02 = 0.008$$

$$P(defective) = 0.006 + 0.008 = 0.014$$

Thus, $P(fromA \mid defective) = \dfrac{P(fromA \cap defective)}{P(defective)} = \dfrac{0.006}{0.006 + 0.008} = \dfrac{3}{7}$

6.

a) The problem is just to find out positions for 4 kinds of molecules from 12 sites, which is to divide the 12 sites into 4 disjoint groups (3 items), using partition formula: $\dfrac{12!}{3!3!3!3!}$=369600

b) If the same kind of molecules cannot be separated, we can consider those three sites together as a *supersite*. In this way, we have 4 supersites and 4 kinds of 3-site group, and the number of arrangements is 4!=24

Hence, we could get the probability: $P = \dfrac{4!}{\frac{12!}{3!3!3!3!}} = \dfrac{24}{369600} = \dfrac{1}{15400}$

(Be conscious about whether question is about the number or probability.)

7.

It's similar with problem5.

a. $P_1 = \dfrac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.02} = \dfrac{11}{233}$ (0r, 0.047)

b. $P_2 = \dfrac{0.001 \times 0.99^2}{0.001 \times 0.99^2 + 0.999 \times 0.02^2} = \dfrac{363}{511}$ (Or, 0.71)

8

<u>Change the 6 defective items to 2 defective items.</u>

By selecting k random samples, passing this batch occurs only when all the selected samples are tested to be non-defective.

The probability of passing it should be:

$$P = \frac{^{198}P_k}{^{200}P_k} = \frac{\frac{198!}{(198-k)!}}{\frac{200!}{(200-k)!}} = \frac{(200-k)(199-k)}{200\times 199} < 0.1$$

$$(200-k)(199-k) < 0.1\times 200\times 199$$

$$(200-k)(199-k) < 3980$$

$$k^2 - 399k + 35820 < 0$$

Using the standard quadratic formula, we have the roots as k = 136.41 or 262.59.

$$136.41 < k < 262.59$$

The minimum required value of k is 137.

9

\# of elements in{A-Z}: 26

\# of elements in{a-z}: 26

\# of elements in{0-9}: 10

\# of elements in {#$%*&@}: 6

Considering special characters, there are 68 choices for each character in the password; Without special character, there are 62 choices.

a. In a password, the ordering of each character does matter. So it is a permutation problem.

When repetition of any character is not allowed:

The total permutations: $^{68}P_6$; Permutations without any special character: $^{62}P_6$

Permutations that there is at least one special character:

$$^{68}P_6 - {}^{62}P_6 = \frac{68!}{62!} - \frac{62!}{56!} = 34,544,754,000$$

b. When repetition is allowed, every character of the password has 68 choices {A-Z,a-z,0-9,special character} and if there is not any special character in the password, every character has 62 choices. So if there is at least one special character, it should be:

$$68^6 - 62^6 = 42,067,247,040$$

10

a. $P = \frac{10^6}{10^{16}} = 10^{-10}$

b. $P = 20\% \times \frac{1}{12} = \frac{1}{60}$

11

a. If only take the first step of redox reaction, there are 4 possible experiments
If take the first and second steps of reaction, there are 4*3=12 experiments
If take all the three steps, there are 4*3*3=36 possible experiments

Total possible experiments: $number = 4 + 4\times 3 + 4\times 3\times 3 = 52$

b. $P = \frac{36}{52} = \frac{9}{13}$

c. No, the result won't change if 5,6 or 7 different amounts are used in the first step.
Assume n different amounts are used in the first step, the result in (b) will be:

$$P = \frac{3\times 3n}{n + 3n + 3\times 3n} = \frac{9n}{13n} = \frac{9}{13}$$

12

a. $P = 0.0008^2 = 6.4\times 10^{-7}$

b. The complement set of at least one drive fails is that both drives don't fail

$$P = 1 - 0.992^2 = 1.6\times 10^{-3}$$

13

a. This is a total probability problem.
If the symbol is 0, the probability of this symbol received correctly is $(1-\varepsilon_0)$;
If the symbol is 1, the probability of this symbol received correctly is $(1-\varepsilon_1)$
So the total probability of a symbol received correctly should be:

$$P = P(0)(1-\varepsilon_0) + P(1)(1-\varepsilon_1) = p(1-\varepsilon_0) + (1-p)(1-\varepsilon_1)$$

b. For the string of 0110, there are two symbol 0s and two symbol 1s in this string and the four symbols are in the specific order. Since the probability of 0 or 1 received correctly is already known, the probability of this string received correctly should be

$$P = (1-\varepsilon_0)^2(1-\varepsilon_1)^2$$

c. In order to correctly decode a 1, the transmitted three-symbol string should contain at least two 1s. For the three-symbol string which contains three 1s is correctly decoded, the probability is $P = (1-\varepsilon_1)^3$

For the three-symbol string is correctly decoded which contains two 1s and one 0, there are three kinds of permutation: 011, 101, 110

For each of them, it has one symbol received incorrectly and two symbol received correctly. So the probability is $P = 3\varepsilon_1(1-\varepsilon_1)^2$

Thus, the probability of a 1 is correctly decoded (the transmitted three-symbol string contains at least two 1s) should be $P = (1-\varepsilon_1)^3 + 3\varepsilon_1(1-\varepsilon_1)^2$

d. Use Bayes' rule in this problem

A: the original symbol is 0

B: the received string is 011

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(A)P(B \mid A) + P(A^C)P(B \mid A^C)}$$

$$P(A) = p$$

$$P(A^C) = 1 - p$$

$$P(B \mid A) = (1-\varepsilon_0)\varepsilon_0^2$$

$$P(B \mid A^C) = \varepsilon_1(1-\varepsilon_1)^2$$

Thus, the probability that a symbol was 0 given that the received string is 011 should be

$$P(A \mid B) = \frac{p\varepsilon_0^2(1-\varepsilon_0)}{p\varepsilon_0^2(1-\varepsilon_0) + (1-p)\varepsilon_1(1-\varepsilon_1)^2}$$

14

According to the normalization axiom of probability:

$$\sum_{i=1}^{n} P_i = 1$$

In this problem

$$P(E_n) = \frac{e^{-\beta E_n}}{Z}$$

The sum of probability is

$$\sum_{n=0}^{\infty} P(E_n) = \sum_{n=0}^{\infty} \frac{e^{-\beta E_n}}{Z} = \frac{1}{Z}\sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+\frac{1}{2})} = 1$$

Equation 1

$$Z = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+\frac{1}{2})} = e^{-\frac{\beta\hbar\omega}{2}} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n}$$

The summation is simply an infinite geometric series with an initial value of 1 and a ratio of $e^{-\beta\hbar\omega}$. Using the formula for an infinite geometric series, we have

$$Z = e^{-\frac{1}{2}\beta\hbar\omega} \frac{1}{1 - e^{-\beta\hbar\omega}} = \frac{1}{e^{\frac{\beta\hbar\omega}{2}} - e^{-\frac{\beta\hbar\omega}{2}}}$$